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COMPLEX JACOBI MATRICES AND QUADRATURE RULES

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ABSTRACT. **Given any sequence of orthogonal polynomials, satisfying the three term recurrence relation**

$$xp_n(x) = \beta_{n+1}p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x),\ p_{-1}(x) = 0,\ p_0(x) = 1,$$

with $\beta_n \neq 0,\ n \in \mathbb{N},\ \beta_0 = 1$, an infinite Jacobi matrix can be associated in the following way

$$J = \begin{bmatrix} \alpha_0 & \beta_1 & 0 & \dots \\ \beta_1 & \alpha_1 & \beta_2 & \dots \\ 0 & \beta_2 & \alpha_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In the general case if the sequences $\{\alpha_n\}$ or $\{\beta_n\}$ are complex the associated Jacobi matrix is complex. Under the condition that both sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are uniformly bounded, the associated Jacobi matrix can be understood as a linear operator J acting on ℓ^2, the space of all complex square-summable sequences, where the value of the operator J at the vector x is a product of an infinite vector x and an infinite matrix J in the matrix sense. The case when the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are not uniformly bounded, an operator acting on ℓ^2 can not be defined that easily. Additional properties of the sequence of orthogonal polynomials are needed in order to be able to define the operator uniquely.

The case when the sequences α_n and β_n are real is very well understood. The spectra of the Jacobi matrix J equals the support of the measure of orthogonality for the given sequence of orthogonal polynomials. All zeros of orthogonal polynomials are real, simple and interlace, contained in the convex hull of the spectra of the Jacobi operator associated with the infinite Jacobi matrix J. Every point in $\sigma(J)$ attracts zeros of orthogonal polynomials. An application of orthogonal polynomials is the construction of quadrature rules for the approximation of integration with respect to the measure of orthogonality.

For arbitrary sequences $\{\alpha_n\}$ and $\{\beta_n\}$ the situation is changed dramatically. Zeros of orthogonal polynomials need not be simple; they are not real and they do not necessarily lie in the convex hull of $\sigma(J)$. There is also a little known about convergence results of related quadrature rule.
Only in recent years a connection between complex Jacobi matrices and related orthogonal polynomials is interesting again (see [2]). Studies of complex Jacobi operators should lead to a better understanding of related orthogonal polynomials, but also the study of orthogonal polynomials with the complex Jacobi matrices should put more light on the non-hermitian banded symmetric matrices.
In this lecture some results are given about complex Jacobi matrices and related quadrature rules, and also some interesting examples are presented.

1. Complex Jacobi matrices

It is well-known that given any two sequences of complex numbers $\{\alpha_k\}$ and $\{\beta_k\}$, $\beta_k \neq 0$, a sequence of monic orthogonal polynomials can be defined by the three-term recurrence relation

$$xp_k(x) = \beta_{k+1}p_{k+1}(x) + \alpha_k p_k(x) + \beta_k p_{k-1}(x), \quad p_0(x) = 1,\ p_{-1}(x) = 0, \tag{1.1}$$

which can be interpreted to be orthogonal with respect to the unique linear functional L, acting on the space of all polynomials $\mathcal{P}$, such that

$$L(p_k p_n) = \delta_{n,k}.$$

This statement is the famous Favard theorem which proof can be found in [4].[1]

For any given two sequences $\{\alpha_k\}$ and $\{\beta_k\}$, $\beta_k \neq 0$, the following infinite matrix can be associated

$$J = \begin{bmatrix} \alpha_0 & \beta_1 & 0 & \dots \\ \beta_1 & \alpha_1 & \beta_2 & \dots \\ 0 & \beta_2 & \alpha_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Different operators acting on the Hilbert space of all square-summable complex sequences ℓ^2, with dense domain, can be associated with the matrix J. Here, the usual inner product $\langle x, y\rangle = \sum x_i \overline{y_i}$ is defined.

The first one can be build as a closed extension of operator defined on the linear span of the set $\mathcal{S} = \{e_n |\ n \in \mathbb{N}_{\not{\,}}\}$, where e_n is a vector having 1 at n-th position and zeros elsewhere. The action of J on $\mathcal{S}$ can be identified to be a simple matrix multiplication. Note that the operator is properly

[1] It will be assumed $\beta_0 = 1$, i.e. the linear functional and all its representations are normalized such that $L(1) = 1$.

defined since columns and rows of J belong to ℓ^2, because the matrix J is banded. This operator will be denoted also by J. In the spectral operator theory the minimal requirement for an operator is to be closed. In general, J is not closed, but it is closable. It is enough to note that

$$\textit{if } y^n \to 0 \textit{ and } Jy^n \to v, \textit{ then } v = 0.$$

This means that the operator J can be extended to be closed to some operator (see [2]), with domain given by

$$\mathcal{D}(J_{\min}) = \{y \mid \exists (y^n)_{n\in\mathbb{N}} \subset \mathcal{S} \text{ converging to } y \text{ and } (Jy^n)_{n\in\mathbb{N}} \text{ converging}\}.$$

The operator created from the matrix J in this way will be denoted by $J_{\min}$, where min stands to emphasize that this operator is a closed operator with the smallest dense domain in ℓ^2 containing the set $\mathcal{S}$.

Another operator can be defined to have as a domain the set of all elements from ℓ^2 which under matrix multiplication with J give an element from ℓ^2. This operator will be denoted by $J_{\max}$, and its domain is given by

$$\mathcal{D}(J_{\max}) = \{y \mid\ Jy \in \ell^2\}.$$

It is clear that operator $J_{\max}$ can be interpreted as an extension of the operator $J_{\min}$, since $\mathcal{S} \subset \mathcal{D}(J_{\max})$. It can be shown that operator $J_{\max}$ is closed.

Simple connections between operators $J_{\min}$, $J_{\max}$ and their adjoints exist (see [2]).

Lemma 1.1. *Let J^H denote a matrix obtained by transposing and conjugating elements of the matrix J, then $(J_{\min})^* = (J^H)_{\max}$ and $(J_{\max})^* = (J^H)_{\min}$. The maximal operator $J_{\max}$ is a closed extension of $J_{\min}$.*

Definition 1.1. *The operator $J_{\min}$ is the difference operator or Jacobi operator associated with the matrix J.*

Definition 1.2. *The infinite matrix J is called proper under condition $J_{\min} = J_{\max}$.*

The following lemma (see [2]) proves that a self-adjoint (bounded or not) operator is proper.

Lemma 1.2. *Under the condition $J = J^H$, which is equivalent to the condition α_k, $\beta_k \in \mathbb{R}$, the operator $J_{\min}$ is a self-adjoint and the matrix J is proper.*

It can be easily seen (see [2]) that for bounded operators, in our case when the sequences $\{\alpha_k\}$ and $\{\beta_k\}$ are uniformly bounded, the operators $J_{\min}$ and $J_{\max}$ are the same.

Lemma 1.3. *Under the condition that both sequences $\{\alpha_k\}$ and $\{\beta_k\}$ are uniformly bounded, the matrix J is proper.*

Let $p(x) = \{p_k(x)\}_{k\in\mathbb{N}}$ be the solution of (1.1). By $q(x) = \{q_k(x)\}_{k\in\mathbb{N}}$ we denote the second linearly independent solution of the difference equation (1.1), which satisfies the initial conditions $q_{-1} = -1$ and $q_0 = 0$.

It has been shown only recently that properness of the Jacobi matrix is equivalent to the determinacy, term which is only connected with the associated orthogonal polynomials (see [3]). The following definition has been introduced by Wall (see [25]).

Definition 1.3. *A complex Jacobi matrix is called determinate if at least one of the sequences $p(0)$ or $q(0)$ is not an element of ℓ^2.*

It has been shown in [3] that terms *properness* and *determination* are equivalent.

Theorem 1.1. *A complex Jacobi matrix J is proper if and only if it is determinate.*

It was also proved by Wall (see [25]), that under the condition that the sequences $p(z)$ and $q(z)$ are both in ℓ^2 for some $z \in \mathbb{C}$, then they are both in ℓ^2 for every $z \in \mathbb{C}$. This has interesting consequence on the spectrum of indeterminate Jacobi operators. Since for every $z \in \mathbb{C}$ the sequence $p(z)$ is in ℓ^2, every point z in the complex plane is an eigenvalue of $J_{\max}$ with the eigenvector $p(z)$, i.e., $\sigma(J_{\max}) = \mathbb{C}$.

An interest for the associated Jacobi operator is motivated by results for the self-adjoint operators (bounded or not). It is known that in the case when an associated operator to the matrix J is self-adjoint, all zeros of orthogonal polynomials are simple, real and lie in the convex hull of the spectrum of J. This is not true for the general complex Jacobi operator. However, there are some connections of the zeros of orthogonal polynomials and the spectrum of the associated Jacobi operator. Further, we will consider only proper Jacobi operators. A unique Jacobi operator, with dense domain in ℓ^2, associated with Jacobi matrix J will be denoted again by J.

One of the most important results is the one concerned with the zeros of orthogonal polynomials. First, we need some definitions:

Definition 1.4. *The numerical range $\Theta(J)$ of the operator J is defined by*

$$\Theta(J) = \{\langle Jy, y\rangle \mid y \in \mathcal{D}(J),\ ||y|| = 1\}.$$

The closure of $\Theta(J)$ is denoted by $\Gamma(J)$.

Note that for a bounded Jacobi operator, the numerical range and $\Gamma(J)$ are bounded by the norm of the operator, since $|\langle Jy, y\rangle| \le ||Jy||||y|| \le ||J||$.

Definition 1.5. *A Jacobi operator obtained from J by shifting sequences $\{\alpha_k\}$ and $\{\beta_k\}$ for n positions, i.e. from $\alpha_k^n = \alpha_{k+n}$ and $\beta_k^n = \beta_{k+n}$ is denoted by $J^{(n)}$.*

Definition 1.6. (see [2], [8]) *An intersection of* $\Gamma(J^{(n)})$ *is denoted by* $\Gamma_{\text{ess}}(J)$ *and the essential spectrum of* J *by*

$$\sigma_{\text{ess}}(J) = \{z|\ \textit{range of } zI - J \textit{ is not closed}\}.$$

In [2] the following theorem can be found

Theorem 1.2. (a) *There are no zeros of polynomials* $\{p_n\}$ *outside of* $\Gamma(J)$.

(b) *There holds* $\Gamma_{\text{ess}}(J) \subset \Gamma(J^{(n+1)}) \subset \Gamma(J^{(n)})$ *for* $n \in \mathbb{N}_{\not\nu}$.

(c) $\sigma(J) \subset \Gamma(J)$ *and* $\sigma_{\text{ess}}(J) \subset \Gamma_{\text{ess}}(J)$, *the set* $\sigma(J)\backslash\Gamma_{\text{ess}}(J)$ *consists of isolated points which accumulate only on* $\Gamma_{\text{ess}}(J)$.

In the last section of this paper we show that the bound $\Gamma(J)$ for the zeros of orthogonal polynomials can not be significantly improved in a general case.

In the case of the bounded Jacobi operator J, the point spectrum can be obtained from the Weyl function, where this function is defined by (see [2])

$$\phi(z) = \langle (zI - J)^{-1} e_0, e_0 \rangle, \quad z \in \mathbb{C}\backslash\sigma(J).$$

It is known that the Weyl function is analytic on its domain, and it is not on any larger set. In [1] the following theorem is given:

Theorem 1.3. *Let* ζ *be isolated point of* $\sigma(J)$. *Then* $\zeta \in \sigma_{\text{ess}}(J)$ *if and only if* ϕ *has essential singularity in* ζ, *and* ζ *is an eigenvalue of algebraic multiplicity* $m < +\infty$ *if and only if* ϕ *has a pole of multiplicity* m. *In particular, if* $\sigma(J)$ *is countable, the set of singularities of* ϕ *coincides with* $\sigma(J)$.

This means that from the Weyl function spectrum of a compact Jacobi operator can be recovered completely.

Under condition $|z| > ||J||$, the operator $(zI - J)^{-1}$ can be expressed in the following form (see [22], [8])

$$(zI - J)^{-1} = \frac{1}{z} \sum_{k=0}^{+\infty} (z^{-1} J)^k.$$

Using the previous formula the Laurent expansion around infinity for the Weyl function can be given by

$$\phi(z) = \frac{1}{z} \sum_{k=0}^{+\infty} \frac{\langle J^k e_0, e_0 \rangle}{z^k}.$$

Using the Cauchy integral formula a complex valued measure of orthogonality μ can be reconstructed from the Weyl function (see [2], [23]), by means (2.3)

$$\oint_C z^\nu p_n(z) \phi(z)\, dz = 0, \quad \nu \in \{0, 1, \dots, n-1\},$$

where the contour C is chosen to be in the set $\{z|\ |z| > ||J||\}$ and to wind around $\sigma(J)$ once.

In fact, given some complex measure with a compact support μ, the linear functional L acting on the space of all polynomials $\mathcal{P}$ can be introduced by

$$L(p) = \int p(x) d\mu(x), \quad p \in \mathcal{P},$$

under condition that all moments exist (see Eq. (2.1)). From the measure μ, the Weyl function for the associated Jacobi operator, can be reconstructed on the set $U = \{z| \ |z| > ||J||\}$, i.e.

$$\phi(z) = \int \frac{d\mu(x)}{z - x}, \quad z \in U.$$

This is obvious since the quantities $\langle J^k e_0, e_0 \rangle$ are the same as moments of the functional L, and therefore two sides of the previous equation are equal since they have the same Laurent expansions at infinity.

It is important here to note that in the case of complex Jacobi operators, the support of the measure μ need not be equal to the spectrum $\sigma(J)$ of the associated Jacobi operator.

2. Convergence of the quadrature rules

Suppose some measure μ is given, for which set of all polynomials $\mathcal{P}$ are integrable, i.e. for which all integrals of the form

$$\mu_k = \int x^k d\mu(x), \tag{2.1}$$

do exist. The sequence μ_k is called moments of the measure μ. For the measure μ a linear functional defined on the space of all polynomials $\mathcal{P}$, can be associated in the following way

$$L(p) = \int p(x) d\mu(x).$$

Under condition all Hankel determinants $\det(\mu_{i+j})_{i,j=0,\ldots,n}$ are different from zero[2], the sequence of orthogonal polynomials with respect to L do exist uniquely (for references see [4]).

Conditions under which zeros of orthogonal polynomials are simple, for a general complex measure, are not known. Hence, in a construction of quadrature rules, the multiplicity of zeros of related orthogonal polynomials has to be introduced. Our quadrature rule in general has the following form

$$G_n(f) = \sum_{k=1}^{N} \sum_{i=0}^{M_k} w_{i,k}^n f^{(i)}(x_k^n).$$

[2]Such linear functional is said to be regular.

Assuming all zeros of the orthogonal polynomials are simple, quadrature rule with respect to the measure μ is the standard Gaussian quadrature rule

$$G_n(f) = \sum_{k=1}^{n} w_k^n f(x_k^n).$$

In both cases, for quadrature rules with multiple and simple nodes, the nodes $x_k^n,\ k = 1, \dots, N$, and the weights $w_{i,k}^n,\ k = 1, \dots, N,\ i = 0, \dots, M_k$, are chosen such that

$$G_n(p) = \int p(x) d\mu(x), \quad p \in \mathcal{P}_{2n-1},$$

where $\mathcal{P}_{2n-1}$ denotes the set of all polynomials of degree at most $2n-1$.

It is well-known (see [13]) that a quadrature rule with simple nodes has the maximal degree of exactness $2n-1$ if and only if the nodes are zeros of the orthogonal polynomial p_n and the weights are solutions of the following system of linear equations

$$\sum_{k=1}^{n} w_k^n (x_k^n)^\nu = \mu_\nu, \quad \nu = 0, 1, \dots, n-1.$$

Since G_n is exact on the space $\mathcal{P}_{2n-1}$, convergence of the sequence $G_n(f)$ can be expected for some suitable f.

An ideal case is when zeros of orthogonal polynomials are all contained in the support of measure μ. In that case, the conditions on function f are the weakest.

Example 2.1. For Jacobi measure

$$d\mu(x) = \chi_{[-1,1]}(1-x)^\alpha (1+x)^\beta dx,$$

the all zeros of orthogonal polynomials are contained in the support of the measure μ, and it can be proved that Gaussian quadrature rules are converging for any continuous function on $[-1, 1]$ (see [13]).

When zeros of orthogonal polynomials are not contained in the support of measure μ, the convergence of Gaussian quadrature rules can not be expected for all continuous functions on the support of μ.

Example 2.2. The simplest example which can be considered is the one where the measure μ is characteristic function of the set $[-2,-1] \cup [1,2]$. This measure μ is positive, hence, related Jacobi operator is self-adjoint (and bounded) and the spectrum of Jacobi operator coincides with the supporting set of the measure μ. However, it can be easily checked that zeros of orthogonal polynomials are not all contained in the supporting set. Since the measure is symmetric, it is easy to see that for all polynomials of odd degree ($p_{2k+1}(0) = 0,\ k \in \mathbb{N}_{\not{0}}$) and number zero is not contained in the support of μ.

It is obvious that in this case, not only the continuity on the supporting set of the measure is needed for convergence of $G_n(f)$, but also some additional conditions on f are needed in order to assure the convergence of quadrature rules[3].

Under the condition that zeros of orthogonal polynomials are localized, which is true for bounded Jacobi operator J (see Theorem 1.2), in some bounded domain (called $\Gamma(J)$), it can be easily checked that G_n, with multiple nodes, is exact for all $p \in \mathcal{P}_{2n-1}$, under assumption the nodes x_k^n and the weights $w_{i,k}^n$ are chosen to be

$$\frac{Q_n}{P_n} = \sum_{k=1}^{N} \sum_{i=0}^{M_k} \frac{i! w_{i,k}^n}{(z - x_k^n)^{j+1}},$$

i.e. provided they are obtained from the fractional decomposition of quotient of the monic numerator polynomial Q_n and monic orthogonal polynomial P_n.[4]

In order to see this fact, we consider an integral of the form

$$\oint_C \left(\phi(z) - \frac{Q_n}{P_n} \right) p(z)\, dz, \quad p \in \mathcal{P}_{2n-1}, \quad \Gamma(J) \subset \operatorname{int}(C), \tag{2.2}$$

where ϕ is the Weyl function, P_n and Q_n are orthogonal and numerator polynomials associated to Jacobi operator J. According to Theorem 1.2 it can be easily seen that all zeros of P_n are localized in $\operatorname{int}(C)$ and our integrand is analytic in the domain $\mathbf{C} \backslash \operatorname{int}(C)$.

Applying Cauchy's theorem on the second integral in (2.2) and using partial fraction decomposition, we get

$$\oint_C \frac{Q_n(z)}{P_n(z)} p(z)\, dz = \sum_{k=1}^{N} \sum_{i=0}^{M_k} w_{i,k}^n p^{(i)}(x_k^n).$$

On the other hand, since the Weyl function coincides with

$$\int \frac{d\mu(x)}{z - x}, \quad |z| > ||J||,$$

using Fubini's theorem (see [17]), we have

$$\oint_C \phi p dz = \oint_C p(z) dz \int \frac{d\mu(x)}{z - x} = \int d\mu(x) \oint_C \frac{p(z)}{z - x} dz = \int p d\mu.$$

[3]Such zeros of orthogonal polynomials which are not connected with the spectrum of the related Jacobi operator are called spurious (see [23]).

[4]Monic numerator polynomials associated to some Jacobi operator J are monic orthogonal polynomials associated to Jacobi operator $J^{(1)}$ (cf. Definition 1.5 and [4], [2])

It is well-known, an estimate from Padé approximation theory (see [9], [2], [23]) gives

$$\phi - \frac{Q_n}{P_n} = O(z^{-2n-1}), \quad z \to \infty.$$

Thus, the integrand in the (2.2) has an asymptotic behavior near infinity at least z^{-2} for $p \in \mathcal{P}_{2n-1}$, i.e. the integral in (2.2) is equal to zero.

Lemma 2.1. *For any polynomial $p \in \mathcal{P}_{2n-1}$*

$$\int p d\mu = \sum_{k=1}^{N} \sum_{i=0}^{M_k} w_{i,k}^n p^{(i)}(x_k^n).$$

The following lemma holds ([6]):

Lemma 2.2. *For any analytic function f on some set $U \supset \operatorname{int}(C) \supset \Gamma(J)$, we have*

$$\int f d\mu - G_n(f) = \frac{1}{2\pi \mathrm{i}} \oint_C f(z) \left(\phi - \frac{Q_n(z)}{P_n(z)} \right) dz.$$

It can be proved easily since

$$G_n(f) = \oint_C f(z) \frac{Q_n(z)}{P_n(z)} \, dz$$

and rational function Q_n/P_n has only zeros of P_n as singularities in $\operatorname{int}(C)$.

The following expression can be given for $\phi(z) - Q_n(z)/P_n(z)$ (see [6]):

Lemma 2.3. *For $z \in \{z| \; |z| > ||J||\}$, there holds*

$$\phi(z) - \frac{Q_n(z)}{P_n(z)} = \frac{1}{P_n(z)^2} \int \frac{P_n(x)^2}{z-x} d\mu(x). \tag{2.3}$$

Equality (2.3) is obtained by the following argumentation

$$\begin{aligned} (P_n\phi - Q_n)(z) &= \frac{1}{2\pi \mathrm{i}} \oint_C \frac{(P_n\phi - Q_n)(\zeta)}{z - \zeta} d\zeta = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{(P_n\phi)(\zeta)}{z-\zeta} d\zeta \\ &= \frac{1}{P_n(z)} \int \frac{P_n(x)P_n(z)}{z-x} d\mu(x) = \frac{1}{P_n(z)} \int \frac{P_n(x)^2}{z-x} d\mu(x), \end{aligned}$$

where the orthogonality of P_n to $(P_n(z) - P_n(x))/(z-x)$ is used.

Finally everything is set to give an error bound for the integrand analytic in some domain $U \supset \Gamma(J)$.

Theorem 2.1. *Let f be an analytic function in some set $U \supset \operatorname{int}(C) \supset \Gamma(J)$. Then the quadrature rule has the following error*

$$\left| \int f d\mu - G_n(f) \right| \leq \frac{1}{2\pi} \ell(C) \, ||f||_C \left|\left| \phi - \frac{Q_n}{P_n} \right|\right|_C, \tag{2.4}$$

where $||f||_C$ denotes sup norm on the compact set C, and $\ell(C)$ is the length of the curve C.

Theorem 2.2. *For the last term appearing on the right in* (2.4) *the following estimate*

$$\left\|\phi - \frac{P_n}{Q_n}\right\|_C \le \frac{\int |P_n|^2 d|\mu|}{d \min\limits_{z\in C} |P_n(z)|^2} \tag{2.5}$$

holds, where $d = \operatorname{dist}(C, \operatorname{supp}(\mu))$ *denotes distance between curve* C *and the supporting set of the measure* μ.

Since all other values are bounded and constant for a given function f, the only term which determines the convergence is the one appearing in (2.5).

Assuming the associated Jacobi operator is compact it is easy to check using Poincare theorem (see [11]), that

$$\lim_{n\to+\infty} |P_n(z)|^{1/n} = |z|,$$

for z sufficiently large, since ratio of two monic polynomials P_{n+1}/P_n has to be z around infinity. Then, the error term can be bounded by

$$\overline{\lim_{n\to+\infty}} \left\|\phi - \frac{P_n}{Q_n}\right\|_C^{1/2n} \le \frac{1}{\min\limits_{z\in C} |z|} \left(\overline{\lim_{n\to+\infty}} \left(\int |P_n|^2 d|\mu|\right)^{1/2n}\right).$$

Thus, for compact Jacobi operators, quadrature rules are convergent provided the right hand side is smaller then 1. Note that right hand side depends only on the measure μ, which means the convergence of quadrature rules is strictly connected to the properties of μ.

Also, under condition Jacobi operator is the compact perturbation of Jacobi operator associated with the constant sequences

$$\alpha_k = 0, \quad \beta_k = 1/2,$$

an application of Poincare theorem leads to asymptotic expression

$$\lim_{n\to+\infty} |P_n(z)|^{1/n} = \left|\frac{z+\sqrt{z^2-1}}{2}\right|,$$

and the bound is given by

$$\overline{\lim_{n\to+\infty}} \left\|\phi - \frac{P_n}{Q_n}\right\|_C^{1/2n} \le \frac{4}{\min\limits_{z\in C} |z+\sqrt{z^2-1}|}\ \overline{\lim_{n\to+\infty}} \left(\int |p_n|^2 d|\mu|\right)^{1/2n},$$

where p_n is the orthonormal polynomial with respect to μ. Again the convergence is assured in the case when right hand side is smaller then 1. The same comment as for compact Jacobi operators applies here, too.

In the case integrals with respect to $|\mu|$ of $|p_n|^2$ are uniformly bounded, the convergence is assured. This is true for Magnus class of complex measures (see [9]):

Theorem 2.3. *Let w be the complex function on $[-1,1]$ such that there is real function ω, integrable and positive almost everywhere on $[-1,1]$ such that w/ω is continuous and non vanishing on $[-1,1]$. Then*

(a) *when n is large enough, the orthogonal polynomial p_n has exact degree n and is uniquely determined, up to a multiplicative constant, and*

$$\int_{-1}^{1} p_n^2(x)w(x)dx \neq 0.$$

(b) *following integrals are uniformly bounded $\int_{-1}^{1} |p_n(x)|^2|w(x)|dx \leq C$.*

(c) *when n is large enough, a three term recurrence relation*

$$xp_n = \beta_{n+1}p_{n+1} + \alpha_n p_n + \beta_n p_{n-1},$$

holds, with condition $\lim_{n\to+\infty} \alpha_n = 0$ and $\lim_{n\to+\infty} \beta_n = 1/2$.

In the last section an example is presented which is a good candidate for an extension of the previous theorem, i.e. the complex measure is presented which vanishes (at one point) on the interval $[-1,1]$ and still, as numerical examples show, Jacobi operator is bounded and related orthogonal polynomials have bounded and simple zeros.[5]

Note that the convergence of the quadrature rules can be measured by calculation of the integrals

$$\int |P_n|^2 d|\mu|.$$

Smaller the value of this integral smaller will be the error in the quadrature rule. For calculation of this integral, Gaussian quadrature rules for positive measure μ can be used (see [12]).

Before Magnus presented his result in [9], there was result concerning measures of the form $rd\mu$ (see [10]), where the positive measure μ is a compact perturbation of the Chebyshev measure, or equivalently μ belongs to the class $\mathcal{M}[0,1]$ (see [24]) and r is a complex rational function having zeros and singularities outside $[-1,1]$. However, that result is now contained in the Magnus theorem.

It seams, that the first paper concerning Gaussian quadrature rules for the complex weight functions was given by Nuttall and Wherry [20]. If $f(\theta) = w(\cos\theta)|\sin\theta|$, Nuttal and Wherry proved that Gaussian quadrature rules are converging for the weight functions w contained in any of the following two classes:

Class 1. The function f is integrable on $(-\pi,\pi)$, and if h_k are Fourier

[5]It is important to note the work of H. Stahl [23], who has shown that for weight function $\chi_{[-1,1]}(x-\cos(\pi\alpha_1))(x-\cos(\pi\alpha_2))/\sqrt{1-x^2}$, where 1, α_1 and α_2 are irrational numbers, algebraically independent, associated Jacobi operator is unbounded and zeros of related orthogonal polynomials cluster everywhere in $\mathbb{C}$.

coefficients of $\log f$, then $\sum_{k\in\mathbf{Z}} |h_k| < +\infty$.

Class 2. On $(-\pi, \pi)$, f satisfies:

- exist A, B such that $A > |f(\theta)| > B > 0$,
- exist $L, \lambda > 0$, independent of θ such that

$$|f(\theta+\delta) - f(\theta)| < L(\log\delta)^{-\lambda-1}.$$

Theorem 2.4. *If w is in one of two classes then asymptotically the zeros of orthogonal polynomials are simple and Gaussian quadrature rules are converging for an analytic integrand in the domain $|z| \le \rho$, $\rho > 1$.*

3. Examples of complex Jacobi operators

3.1. **An example of compact Jacobi operator.** We consider complex compact Jacobi operator. In [15] It was proved that polynomial orthogonal with respect to the linear functional (see [15])

$$L_a^{p,q}(f) = \frac{p-1}{p}\sum_{k=0}^{+\infty}\frac{1}{p^k} f\left(\frac{a}{q^k}\right),$$

exist, if $|p| \ge 1$, $|q| > 1$ or $|p| > 1$, $|q| \ge 1$, where in addition when $|p| = 1$ it is assumed that there exists an $n \in \mathbb{N}$ such that $p^n = 1$, $p \ne 1$, and when $|q| = 1$ it is assumed that for all $n \in \mathbb{N}$, $q^n \neq 1$.[6]

Theorem 3.1. *Polynomials orthogonal with respect to $L_a^{p,q}$, under conditions mentioned before, do exist and they satisfy the following three-term recurrence relation*

$$xp_k(x) = \beta_{k+1}p_{k+1}(x) + \alpha_k p_k(x) + \beta_k p_{k-1}(x),$$

where

$$\begin{aligned}
\alpha_k &= aq^k\frac{p+q-2pq^k(1+q)+pq^{2k}(p+q)}{(pq^{2k-1}-1)(pq^{2k+1}-1)}, \quad k \ge 0,\\
\beta_k^2 &= a^2p\,q^{2k}\frac{(q^k-1)^2(pq^{k-1}-1)^2}{(pq^{2k-2}-1)(pq^{2k-1}-1)^2(pq^{2k}-1)}, \quad k \ge 1.
\end{aligned}$$

The sequence of moments is given by

$$\mu_k = \frac{(p-1)(aq)^k}{pq^k-1}.$$

It is easy to check that associated Jacobi operator (denoted by $J_a^{p,q}$) is compact, except for $|q| = 1$ when it is bounded. Bounds for zeros can be obtained from the following theorem (see [15]):

[6]Note that positive definite case appears under condition $p > 1$ and $q > 1$, corresponding orthogonal polynomials are known as little $1/q$ Jacobi polynomials (see [7]).

Theorem 3.2. *Under condition $|q| > 1$ the linear operator $J_a^{p,q}$, is compact, even more it is of trace class. In the case when $|q| = 1$, with $q^n \neq 1,\ n \in \mathbb{N}$, and $|p| > 1$, the linear operator $J_a^{p,q}$ is bounded but not compact. All zeros of corresponding orthogonal polynomials lie in the set*

$$\Gamma(J_a^{p,q}) \subset \{z \,|\, |z| \leq |\beta_1| + |\alpha_0|\},$$

provided $|q| > 1$. In the second case all zeros of orthogonal polynomials lie in the set

$$\Gamma(J_a^{p,q}) \subset \left\{ z \;\middle|\; |z| \leq |a| \frac{|p|^2 + 6|p| + 1 + 4\sqrt{|p|}(|p|-1)}{(|p|-1)^2} \right\}.$$

The Weyl function can be reconstructed (since $J_a^{p,q}$ is bounded) from the following equation

$$L_a^{p,q}\left(\frac{1}{z-.}\right) = \frac{p-1}{p} \sum_{k=0}^{+\infty} \frac{1}{p^k} \frac{1}{z - a/q^k} = \sum_{k=0}^{+\infty} \frac{1}{z^{k+1}} \frac{(p-1)(aq)^k}{pq^k - 1},$$

for $|z| > ||J_a^{p,q}||$. The function

$$\int \frac{d\mu(x)}{z-x} = L_a^{p,q}\left(\frac{1}{z-.}\right) = \frac{p-1}{p} \sum_{k=0}^{+\infty} \frac{1}{p^k} \frac{1}{z - a/q^k},$$

under condition $|q| > 1$, is analytic except for the singularities contained in the set $\overline{\{a/q^k|\ k \in \mathbb{N}_{\not{0}}\}}$. It is known that previous function has singularities in the set $\mathrm{supp}(\mu) \cup \sigma(J_a^{p,q})$ (see [2]). The following result can be stated:

Theorem 3.3. *Under condition $|q| > 1$*

$$\sigma(J_a^{p,q}) \subset \overline{\{a/q^k|\ k \in \mathbb{N}_{\not{0}}\}}.$$

When $|q| = 1$ with $q^n \neq 1, n \in \mathbb{N}$, the situation is quite different. Note that q has the representation $\exp(\mathrm{i}b\pi)$ for some irrational b and

$$\int \frac{d\mu(x)}{z-x} = \frac{p-1}{p} \sum_{k=0}^{+\infty} \frac{1}{p^k} \frac{1}{z - a/q^k}.$$

In this case the function has singularities on the set $\{z|\ |z| = |a|\}$, since the set $\{a \exp(\mathrm{ikb}\pi)|\ \mathrm{k} \in \mathbb{N}_{\not{0}},\ \ \text{irrational}\}$ is dense in $\{z|\ |z| = |a|\}$ (see [15]). The same is true for the Weyl function since two functions coincide on the set $\{z|\ |z| > |a|\}$. The following theorem can be stated:

Theorem 3.4. *Under condition $|q| = 1,\ q^n \neq 1,\ n \in \mathbb{N}$,*

$$\{z|\ |z| = |a|\} \subset \sigma(J_a^{p,q}).$$

Note that in this case, the bound for $\Gamma(J_a^{p,q})$, given in Theorem 3.2, for $|p|$ sufficiently large tends to $|a|$. Hence, $\Gamma(J_a^{p,q})$ is shrinking around circle $|z| = |a|$, but cannot go beyond. This means that sharper bound for location of zeros then it is given in Theorem 1.2 cannot be given in a general case. Note that here the support of the measure μ is the subset of the $\sigma(J_a^{p,q})$.[7]

For construction of quadrature rules we used usually QR-algorithm. It turns out it is stable. For extremely big number of nodes, the construction is stable for all values of p and q. A bifurcation phenomenon, encountered in the paper [21] for the generalized Bessel polynomials, has not been detected even in the cases when $|p| = 1$ or $|q| = 1$.

For the construction of the weights of related quadrature rules, a method suggested in [5] has been used, since the measure is discrete and the suggested algorithm in [14] is unstable.

Since for $|p| = 1$, there exists n such that $p^n = 1, p \neq 1$, the functional over polynomial space sums in Cesaro sense (see [18]), the following sums

$$\sum_{k=0}^{+\infty} \mathrm{exp(ik\pi/n)f(a/q^k)}, \quad \mathrm{n} \in \mathbb{N},$$

with $\lim\limits_{x \to 0} f(x) < +\infty$, can be evaluated. For example, the sum can be found

$$\sum_{k=0}^{+\infty} (-1)^k \exp(a/q^k).$$

Under assumption that zeros of the orthogonal polynomials are simple, Gaussian quadrature rules can be applied and machine precision is achieved with only 10 nodes, for example, if $q = 2$.

It is interesting that even for q very close to 1, the convergence is very fast (see an example with positive measure in [15]). Many examples have been run in all of them, and we were unable to find a case when polynomial has zero of higher multiplicity then 1. It turns out that orthogonal polynomials does not have spurious zeros; all zeros are attracted to the supporting set of linear functional.

For $|q| = 1,\ q^n \neq 1,\ n \in \mathbb{N},\ |\imath| > \nVdash$, similar results like with $|p| = 1$ can be encountered. In many cases we were unable to encounter a case when zeros of orthogonal polynomials have higher multiplicity then 1. It turns out again that all zeros are attracted to the supporting set of measure μ.

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[7]Note that there are sequences of orthogonal polynomials for which associated Jacobi operator has spectrum $\sigma(J) = \{0\}$, which can be the case for compact operators (see [8], [22]). In this case, the orthogonality measure can be reconstructed on some contour surrounding zero but obviously spectrum cannot be larger then supporting set of the measure. This is the case, e.g. with generalized Bessel polynomials (see [4]), there is also beautiful example given in [1].

For example, for getting the following sum

$$\sum_{k=0}^{+\infty}\left(-\frac{10}{11}\right)^k \exp(\exp(-\mathrm{i}\mathrm{k}\sqrt{2}\pi)),$$

with machine precision we need only 10 nodes in the corresponding Gaussian quadrature rule.

3.2. **'Bounded' Jacobi operator.** Another class of orthogonal polynomials has been considered in [16]. Polynomials orthogonal with respect to the weight function

$$w(x) = \chi_{[-1,1]} m\pi \mathrm{i}(-1)^{\mathrm{m}}\mathrm{x}\exp(\mathrm{i}\mathrm{m}\pi\mathrm{x})/2, \quad \mathrm{m} \in \mathbb{Z}\backslash\{0\}.$$

The following theorems can be found in [16].

Theorem 3.5. *For every integer $m\,(\neq 0)$, the sequence of orthogonal polynomials with respect to the weight function $w(x) = x\exp(\mathrm{i}\mathrm{m}\pi\mathrm{x})$, supported on the interval $[-1,1]$, does exist.*

Theorem 3.6. *Polynomials orthogonal with respect to w satisfy the following differentially-difference equation*

$$-x\phi p_n' = p_2^n p_n + q_2^n p_{n-1},\ \phi = 1 - x^2,\ \deg(p_2^n) = 2,\ \deg(q_2^n) = 2.$$

Theorem 3.7. *Under condition that p_n and q_2^n do not have common zeros, zeros of p_n are simple. In the case when zeros of q_2^n are simple, the polynomial p_n can have at most two zeros of the second order, which are negative conjugates to each other. If a zero of q_2^n is double for odd n, p_n can have zero of third order on the imaginary axis, and for even n, p_n can have zero of third order on the imaginary axis in which case it has another simple zero on the imaginary axis, too.*

In all numerical calculations it turns out that all zeros of the orthogonal polynomials are simple, which means last theorem is not sharp enough.

A significance of the measure $\chi_{[-1,1]}(-1)^m m\pi\mathrm{i}\mathrm{x}\exp(\mathrm{i}\mathrm{m}\pi\mathrm{x})/2$ lies in the following simple fact that, as numerical examples shows, it should be the first positive example of the complex weight having value zero inside the supporting set and yet the associated Jacobi operator is bounded (see Theorem 2.3). There is also an important application of this measure in the construction of Gaussian quadrature rules which can be applied for calculating integrals having highly oscillatory integrands.

Note, also, that the weight function $\chi_{[-1,1]}\exp(\mathrm{i}\mathrm{m}\pi\mathrm{x})$ is contained in the Magnus class (see Theorem 2.3), however it is simple to check that $\beta_0 = 0$, i.e. the orthogonal polynomials exist but only asymptotically. The existence problem which is evident for the weight function $\chi_{[-1,1]}\exp(\mathrm{i}\mathrm{m}\pi\mathrm{x})$, has as a consequence, impossibility of construction of orthogonal polynomials using standard software packages (see [5]).

The sequence of moments for the measure $\chi_{[-1,1]} m\pi \mathrm{ix}\exp(\mathrm{im}\pi\mathrm{x})/2$ is given by

$$\mu_k = \frac{(-1)^k (k+1)!}{2(\mathrm{im}\pi)^{\mathrm{k}}} \sum_{\nu=0}^{k} \frac{(1+(-1)^\nu)(-\mathrm{im}\pi)^\nu}{(\nu+1)!}.$$

It turns out in numerical calculations that three term recurrence coefficients satisfy the conditions

$$\lim_{k\to+\infty} \alpha_k = 0, \qquad \lim_{n\to+\infty} \beta_k = \frac{1}{2}.$$

Having as a consequence the proof of convergence of quadrature rules provided that $\int |P_n|^2 |x|\, dx$, is bounded (see Eq. (2.4)).

At the end of this discussion, we consider one numerical example involving oscillatory integrands. Namely, we calculate the Fourier coefficients $F_m(f) = C_m(f) + \mathrm{iS_m(f)} = \int_{-1}^{1} \mathrm{f(x)}\exp(\mathrm{im}\pi\mathrm{x})\,\mathrm{dx}$. Since $\int\limits_{-1}^{1} \exp(\mathrm{im}\pi\mathrm{x})\mathrm{dx} = 0$, we have

$$F_m(f) = \int_{-1}^{1} g(x) w(x)\, dx,$$

where $w(x) = x\exp(\mathrm{im}\pi\mathrm{x})$, and

$$g(x) = \frac{f(x) - f(0)}{x} \quad (x \neq 0), \quad g(0) = f'(0).$$

Numerical construction of the Gaussian quadrature rule was investigated in details in [16].

We take an integrand having singularities near interval $[-1,1]$, i.e. $f(x) = x/(x^2+1/4)$.

The Gaussian approximations of the integral $S_m(f) = \int\limits_{-1}^{1} f(x)\sin(m\pi x)dx$ are given in Table 3.1.

n	$S_{10}(f)$
10	-0.0509124802888631
20	-0.0509124798498521
30	-0.0509124699339274
40	-0.0509120078597893
50	-0.0509120064014030
60	-0.0509120064013063

TABLE 3.1. Values of Gaussian approximations for $f(x) = x/(x^2+1/4)$

It is interesting to note that convergence can be significantly accelerated by the knowledge of the value of residuum of the function f at the point $\mathrm{i}/2$, until zeros of the orthogonal polynomials drop below this point. The residuum at this point is given by

$$2\pi \mathrm{i} \operatorname*{Res}_{\mathrm{z}=\mathrm{i}/2} \{\mathrm{f}(\mathrm{z})\mathrm{e}^{\mathrm{i}m\pi \mathrm{z}}\} = \pi \mathrm{i} \exp(-10\pi/2) \approx 4.734434401198 \times 10^{-7}\mathrm{i}.$$

The zero distribution is given in Figure 3.1. It is easy to check that quadrature rules with 10, 20 and 30 nodes are calculating integral on the interval $[-1, 1]$ together with residuum at the point $\mathrm{i}/2$ (simply add value of residuum at $\mathrm{i}/2$ to results of quadrature rules to get better estimates). However, when zeros drop below point $\mathrm{i}/2$ (see Figure 3.1), that is the case for number of nodes equal 40, quadrature rules are converging to the value of the integral on $[-1, 1]$.

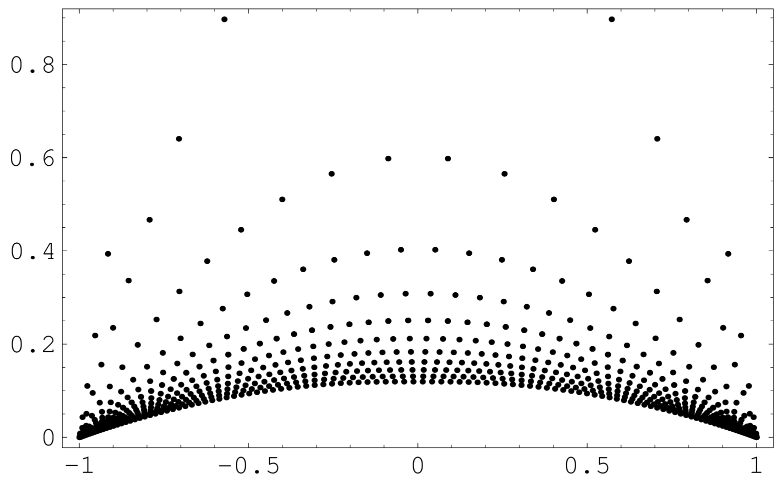

FIGURE 3.1. Distribution of zeros of orthogonal polynomials of degrees $10, \dots, 120$ with step 10, for $m = 10$.

Numerous other examples were presented in [16].

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