Criteria for sets of scores with prescribed positions

Momčilo Bjelica and Slobodan Lakić

Abstract

We characterize sets of scores with property that each score has prescribed position in the tournament score sequence[1].

A tournament T_n is a graph with vertices $1, 2, \ldots, n$ such that each pair of distinct vertices i and j is joined by one and only one of the oriented edges ij and ji. We say that vertex i dominates vertex j if T_n contains an oriented edge ij. The score (outdegree) of vertex i is the number s_i of vertices that i dominates. Let vertices of T_n be labeled in such way that $s_1 \leq s_2 \leq \ldots \leq s_n$. The sequence $(s_1, s_2, \ldots, s_n)$ is called the score sequence of T_n. A transitive tournament has the simplest structure $0 \leq 1 \leq \ldots \leq n-1$, while regular tournaments have scores as nearly equal as possible $\underbrace{\lfloor e \rfloor = \ldots = \lfloor e \rfloor}_{\lfloor n/2 \rfloor} \leq \underbrace{\lceil e \rceil = \ldots = \lceil e \rceil}_{\lceil n/2 \rceil}, \quad e = (n-1)/2.$ E.g, a regular tournament of odd order $n = 2e + 1$ obtains if vertex j, where $1 \leq j \leq n$, dominates $j + i$, where $1 \leq i \leq e$, assuming that vertex $n + k$ equals k. Landau theorem [6] gives a non-constructive criterium for a score sequence.

Theorem 1 (Landau theorem) *A nondecreasing sequence $s_1 \leq s_2 \leq \ldots \leq s_n$ of nonnegative integers is the score sequence of some tournament T_n if and only if*

$$\sum_{i=1}^{k} s_i \geq k2, \quad 1 \leq k \leq n, \quad \sum_{i=1}^{n} s_i = n2. \tag{1}$$

The next theorem [2] gives a criterium for score segments and subsequences with arbitrary positions of scores.

Theorem 2 *Let $t_1 \leq t_2 \leq \ldots \leq t_m$ be a sequence of nonnegative integers and $s_1 \leq s_2 \leq \ldots \leq s_n$ be a score sequence of a tournament T_n with $m \leq n$. Then the following properties are equivalent:* 4 $S_1: \quad \sum_{i=1}^{j} t_i \geq j2, \quad 1 \leq j \leq m;$

$S_2: \quad t_j = s_j, \quad 1 \leq j \leq m, \quad for some\, T_n;$

$S_3: \quad t_j = s_{k+j}, \quad 1 \leq j \leq m, \quad for some\, T_n$ *and* k*;*

$S_4: \quad t_j = s_{k_j}, \quad 1 \leq j \leq m, \quad for some\, T_n$ *and* $k_1 < k_2 < \ldots < k_m$.

[1]Presented at the IMC "Filomat 2001", Niš, August 26–30, 2001

2000 Mathematics Subject Classification: 05C20.
Keywords: Tournament, score sequence.

In this paper we consider conditions for a set of integers to be the subset of scores with prescribed positions in some score sequence. In the following we shall use the notations b(x)=x2, X(k)=$\sum_{i=1}^k x_i$, $\sum_{i=l}^k x_i = 0, \quad l > k.$

Theorem 3 *Let $0 \le t_1 \le t_2 \le \ldots \le t_m$ and $0 < k_1 < k_2 < \ldots < k_m$ be two sequences of integers. Then, there exists a tournament T_n with score sequence $s_1 \le s_2 \le \ldots \le s_n$ such that $t_j = s_{k_j}, \quad 1 \le j \le m$, if and only if*

$$\sum_{i=1}^{j}(k_i - k_{i-1})t_i \ge k_j 2, \quad 1 \le j \le m, \ \ k_0 = 0. \tag{2}$$

The size of the tournament can be k_m if and only if in e:prescribed the equality holds for $j = m$.

Necessity. If for some tournament we have $t_j = s_{k_j}$, where $1 \le j \le m$, then monotonicity of the score sequence and the Landau theorem give $\sum_{i=1}^{j}(k_i - k_{i-1})t_i = \sum_{i=1}^{j}(k_i - k_{i-1})s_{k_i} \ge \sum_{i=1}^{k_j} s_i \ge k_j 2, \quad 1 \le j \le m.$

Sufficiency. Let some sequences t and k satisfy e:prescribed. Define the sequence $u_1 \le u_2 \le \ldots \le u_{k_m}$ which includes the sequence t as subsequence u$_k = t_j, \quad k_{j-1} < k \le k_j, \quad 1 \le j \le m$, and let us prove that it satisfies property S_1 from Theorem 2. In the following minorizations we apply piecewise linearity of U, inequalities e:prescribed, and convexity of binomial function b U(k)=U(k$_{j-1}$) $+ (k - k_{j-1})t_{k_j}$
$= U(k_{j-1}) + (k - k_{j-1})\frac{U(k_j) - U(k_{j-1})}{k_j - k_{j-1}}$
$= \frac{k_j - k}{k_j - k_{j-1}} U(k_{j-1}) + \frac{k - k_{j-1}}{k_j - k_{j-1}} U(k_j)$
$\ge \frac{k_j - k}{k_j - k_{j-1}} b(k_{j-1}) + \frac{k - k_{j-1}}{k_j - k_{j-1}} b(k_j)$
$\ge b\left(\frac{k_j - k}{k_j - k_{j-1}} k_{j-1} + \frac{k - k_{j-1}}{k_j - k_{j-1}} k_j\right)$
$= b(k)$. By property S_2 from Theorem 2, there exists a tournament T_n with beginning score segment u. Therefore, scores from the sequence t appear on the prescribed positions k.

Hardy, Littlewood and Pólya [4] introduced the *majorization* relation. Let a and b be in $\mathbb{R}^n$, then a is said to be *majorized* by b a$\prec b \quad if \quad \sum_{i=1}^k a_{[i]} \le \sum_{i=1}^k b_{[i]}, \quad 1 \le k \le n, \quad \sum_{i=1}^n a_{[i]} = \sum_{i=1}^n b_{[i]}$, where $x_{[1]} \ge x_{[2]} \ge \ldots \ge x_{[n]}$ denotes the nonincreasing permutation of the sequence $x_1, x_2, \ldots, x_n$. The same authors proved that $a \prec b$ if and only if for each convex function f holds $\sum_{i=1}^n f(a_i) \le \sum_{i=1}^n f(b_i)$. In this notation the Landau condition e:Landau takes an equivalent form

$$(s_1, s_2, \ldots, s_n) \prec (0, 1, \ldots, n-1). \tag{3}$$

Weak supermajorization [7] of a by b means a$\prec^w b \quad if \quad \sum_{i=1}^k a_{(i)} \ge \sum_{i=1}^k b_{(i)}, \quad 1 \le k \le n$, where $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$ is the nondecreasing permutation of $x_1, x_2, \ldots, x_n$, and *weak submajorization* means a$\prec_w b \quad if \quad \sum_{i=1}^k a_{[i]} \le \sum_{i=1}^k b_{[i]}, \quad i.e. \quad \sum_{i=k}^n a_{(i)} \le \sum_{i=k}^n b_{(i)}, \quad 1 \le k \le n.$

In the proof of Theorem 3 we constructed the multiset

$$\{k_1 \bullet t_1, (k_2 - k_1) \bullet t_2, \ldots, (k_m - k_{m-1}) \bullet t_m\}$$

which preserves the nondecreasing arrangement of t $\Delta k \bullet t$: $\underbrace{t_1 = \ldots = t_1}_{k_1} \leq \underbrace{t_2 = \ldots = t_2}_{k_2 - k_1} \leq \ldots \leq \underbrace{t_m = \ldots = t_m}_{k_m - k_{m-1}}$. The notation for multisets $\bullet$ we also use for vectors.

Remark 1 *The characterization e:prescribed from Theorem 3 have a condensed form*

$$(0, 1, \ldots, k_m - 1) \ {}^{\mathrm{W}}\!\succ \Delta k \bullet t.2a \tag{4}$$

From Theorems 2 and 3 one obtains that e:prescribed implies S_1. In the following we shall strengthen this implication.

For sequences a and b let $a \leq b$ denotes $a_k \leq b_k$, where $1 \leq k \leq n$.

If the comparands in relations $\leq$ and $\prec^w$ have different lengths, then we restrict them to the shorter one.

Theorem 4 *Let $l_1 < l_2 < \cdots$ and $k_1 < k_2 < \cdots$ be sequences of positive integers and $t_1 \leq t_2 \leq \cdots$. Then the following are equivalent: 3 R_1 : $l \leq k$,*
R_2 : $\Delta l \bullet t \geq \Delta k \bullet t$, $foreacht$,
R_3 : $\Delta l \bullet t \prec^{\mathrm{W}} \Delta k \bullet t$, $foreacht$.

$R_1 \Rightarrow R_2 \Rightarrow R_3$. This is obvious.

$R_3 \Rightarrow R_1$. Let $p_1 \leq p_2 \leq \cdots$ and $q_1 \leq q_2 \leq \cdots$ be the sequences $\Delta l \bullet t$ and $\Delta k \bullet t$, respectively. For $t_1 = \ldots = t_j < t_{j+1}$ we have $\mathrm{l}_j t_1 = \sum_{i=1}^{l_j} p_i \geq \sum_{i=1}^{l_j} q_i \geq l_j t_1$, so that $l_j \leq k_j$.

Corollary 1 *Let $l_1 < l_2 < \ldots < l_m$ and $k_1 < k_2 < \ldots < k_m$ be sequences of positive integers satisfying $l \leq k$. If $t_j = s_{k_j}$, where $1 \leq j \leq m$, is the score subsequence of some tournament T_n, then $t_j = s^*_{l_j}$, where $1 \leq j \leq m$, for some T^*_q.*

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Faculty of Engineering "Mihajlo Pupin",
Zrenjanin, Yugoslavia
bjelica@zpupin.tf.zr.ac.yu

Faculty of Economy,
Brčko, Bosnia and Hertzegovina