# A new ordering relation on lattices applied to weak congruences

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**Abstract**

Starting from an arbitrary codistributive element $a$ in an algebraic lattice $\mathcal{L}$, a new operation $*_a$ on the underlying set $L$ is defined. This operation determines an ordering relation on $L$. A properties of the new poset are investigated. Some necessary and some sufficient conditions under which it is a lattice are presented[1].

An application of the results to the weak congruence lattice is given. A new characterization of the Congruence Extension Property (CEP) in terms of weak congruences under the new ordering is obtained.

## 1 Introduction

Let $\mathcal{L} = (L, \wedge, \vee)$ be an algebraic lattice (sometimes referred to as to $(L, \leq)$) and $a$ an arbitrary codistributive element in $L$ (satisfying $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$, for every $x, y \in L$.)

The mapping $m_a : L \longmapsto \downarrow a$ defined by $m_a(x) = x \wedge a$ is a complete homomorphism from $L$ to the ideal $\downarrow a$. Since the lattice $L$ is algebraic, the classes of the congruence $\phi_a$ induced by this homomorphism always have the top elements (see e.g. [12]). Let $\overline{x}$ denote the top element of the congruence class $\phi_a$ to which $x$ belongs, i.e., $m_a(x) = m_a(\overline{x})$.

The mapping $f : L \longrightarrow L$ defined by $f(x) = \overline{x}$ is a closure operator on $L$. Moreover,

$$\overline{x \wedge y} = \overline{x} \wedge \overline{y}. \tag{1}$$

A binary operation $*_a$ on the set $L$ is defined by

$$x *_a y = (\overline{x} \wedge y) \vee (x \wedge \overline{y}), \text{ for all } x, y \in L. \tag{2}$$

In [5] it is proved that $(L, *_a)$ is a commutative, idempotent groupoid, satisfying the identity:

$$x *_a (x *_a y) = x *_a y. \tag{3}$$

Nevertheless, the operation $*_a$ is not associative in general.

Next, a binary relation $\leq_{*_a}$ on $L$ is defined in a natural way, using operation $*_a$. For all $x, y \in L$, let

$$x \leq_{*_a} y \quad \text{if and only if} \quad x *_a y = y. \tag{4}$$

The relation $\leq_{*_a}$ is an ordering relation on $L$ (see [5]).

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Some of the notions and notations used throughout the paper are listed in the sequel.

As usual, $\downarrow x$ and $\uparrow x$ denote the principal ideal and principal filter in the lattice $\mathcal{L}$, generated by $x$, respectively.

$\downarrow_{*_a} x = \{y \in L \mid y \leq_{*_a} x\}$ and $\uparrow_{*_a} x = \{y \in L \mid x \leq_{*_a} y\}$ are the principal ideal and principal filter in the poset $(L, \leq_{*_a})$ generated by $a$, respectively.

$[b,c] = \{x \in L \mid b \leq x \leq c\}$ is an interval lattice in the lattice $\mathcal{L}$, and $[b,c]_{*_a} = \{x \in L \mid b \leq_{*_a} x \leq_{*_a} c\}$ is an interval in the poset $(L, \leq_{*_a})$.

**Proposition 1** *[5] Let $\mathcal{L} = (L, \wedge, \vee)$ be an algebraic lattice with a codistributive element $a$ and let $\leq_{*_a}$ be the ordering relation defined by (4). Then:*

(**i**) *$a \leq_{*_a} x$, for all $x \in L$.*

(**ii**) *If $b \leq a$, and $x, y \in [b, \overline{b}]$, then $x *_a y = x \vee y$.*

(**iii**) *If $b \leq a$ and $x, y \in [b, \overline{b}]$, then $x \leq y$ if and only if $x \leq_{*_a} y$.*

(**iv**) *If $b \leq a$, then $([b, \overline{b}]_{*_a}, \leq_{*_a})$ is a lattice, isomorphic to the interval in $\mathcal{L}$ bounded by the same elements.*

(**v**) *$\overline{x \wedge y} = \overline{x} *_a \overline{y} = \overline{x} \wedge \overline{y} = \overline{x *_a y}$.*

(**vi**) *$\overline{x} \leq_{*_a} \overline{y}$ if and only if $\overline{y} \leq \overline{x}$.*

(**vii**) *If $x \leq_{*_a} y$ then $\overline{y} \leq \overline{x}$.*

(**viii**) *$\overline{a} *_a x = \overline{x}$.*

(**ix**) *$\overline{a} \leq_{*_a} x$ if and only if $x = \overline{x}$.*

(**x**) *$L$ is the union of all intervals $\left[b, \overline{b}\right]_{*_a}$, for $b \leq a$.*

(**xi**) *The filter $\uparrow_{*_a} \overline{a}$ is antiisomorphic with the ideal $\downarrow a$.*

(**xii**) *$x \leq_{*_a} \overline{0}$, for all $x \in L$, where $0$ is the bottom element in the lattice $\mathcal{L}$.*

From the previous proposition we deduce that $(L, \leq_{*_a})$ is a bounded poset. The bottom element of the poset under consideration is the element $a$ and the top element is $\overline{0}$, where 0 is the bottom element of the lattice $(L, \leq)$.

In the sequel, we represent the algebra $(L, \wedge, *_a)$, by the two ordering operations, $\leq$ and $\leq_{*_a}$, and consequently, by two Hasse diagrams in finite case.

# 2 Results

Let $(L, \leq)$ be an algebraic lattice, let $a \in L$ be a codistributive element in $L$ and let $(L, \leq_{*_a})$ be the poset defined by (4). Firstly, we prove that the distributivity of a finite lattice $(L, \leq)$ is a sufficient condition for $(L, \leq_{*_a})$ to be a lattice.

**Theorem 1** *If $(L, \wedge, \vee)$ is a finite distributive lattice, then $(L, \leq_{*_a})$ is a lattice.*

**Proof.** We prove that under the distributivity condition, the operation $*_a$ is associative.

$x *_a (y *_a z) = (\overline{x} \wedge ((\overline{y} \wedge z) \vee (y \wedge \overline{z}))) \vee (x \wedge \overline{((\overline{y} \wedge z) \vee (y \wedge \overline{z}))}) =$

$(\overline{x} \wedge \overline{y} \wedge z) \vee (\overline{x} \wedge y \wedge \overline{z}) \vee (x \wedge \overline{y} \wedge \overline{z})$, by (1), (2) and by Proposition 1 (v).

Similarly, we prove that $(x *_a y) *_a z = (\overline{x} \wedge \overline{y} \wedge z) \vee (\overline{x} \wedge y \wedge \overline{z}) \vee (x \wedge \overline{y} \wedge \overline{z})$, and the associativity holds.

By the identities (3): $x *_a (x *_a y) = x *_a y$ and $y *_a (x *_a y) = x *_a y$, we prove that $x *_a y$ is an upper bound of elements $x$ and $y$ under $\leq_{*_a}$. We have to prove that it is the supremum of these elements. Let $p$ be another upper bound of $x$ and $y$, i.e., $x \leq_{*_a} p$ and $y \leq_{*_a} p$. By the definition of $\leq_{*_a}$, $x *_a p = p$ and $y *_a p = p$. Hence, $(x *_a p) *_a (y *_a p) = p$. By the associativity, commutativity and idempotency, $(x *_a y) *_a p = p$, and $x *_a y \leq_{*_a} p$, $x *_a y$ is the supremum. Since every subset has a supremum and there is a bottom element under ordering $\leq_{*_a}$, $(L, \leq_{*_a})$ is a lattice. ∎

Nevertheless, distributivity of $(L, \leq)$ is not a necessary condition for poset $(L, \leq_{*_a})$ to be a lattice, which is illustrated by the following example.

**Example 1** The poset $(L, \leq_{*_a})$ in Figure 1 b) is a lattice, although the lattice $(L, \leq)$ is not distributive.

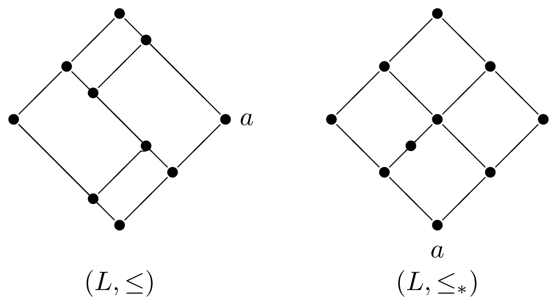


Figure 1 a) Figure 1 b)

Next we prove several lemmas which enable proving a necessary condition under which a poset $(L, \leq_{*_a})$ is a lattice as well as a characterization of the CEP in Section 3.

**Lemma 1** *Let $x, y, z \in L$, such that $x \leq_{*_a} z$ and $y \leq_{*_a} z$. Then:*
**(i)** $\overline{x \wedge y} \leq_{*_a} \overline{z}$.
**(ii)** $x \wedge \overline{y} \leq_{*_a} z$ *and* $\overline{x} \wedge y \leq_{*_a} z$.

**Proof.** **(i)** From $x \leq_{*_a} z$ and $y \leq_{*_a} z$, by Proposition 1 (vii), we have that $\overline{z} \leq \overline{x}$ and $\overline{z} \leq \overline{y}$. Thus, $\overline{z} \leq \overline{x} \wedge \overline{y} = \overline{x \wedge y}$, hence $\overline{x \wedge y} \leq_{*_a} \overline{z}$.
**(ii)** By $(\overline{x} \wedge z) \vee (x \wedge \overline{z}) = z$,

$$(\overline{x} \wedge \overline{y} \wedge z) \vee (x \wedge \overline{y} \wedge \overline{z}) \leq (\overline{x} \wedge z) \vee (x \wedge \overline{z}) = z. \tag{5}$$

From $x \leq_{*_a} z$ and $y \leq_{*_a} z$, by Proposition 1 (vii), $\overline{z} \leq \overline{x \wedge y}$. From $z \leq \overline{x \wedge y} = \overline{x} \wedge \overline{y}$, we have that

$$z \leq (\overline{x} \wedge \overline{y} \wedge z) \vee (x \wedge \overline{y} \wedge \overline{z}). \tag{6}$$

From (5) and (6) we have that $(\overline{x} \wedge \overline{y} \wedge z) \vee (x \wedge \overline{y} \wedge \overline{z}) = z$, i.e.,

$$(\overline{x \wedge \overline{y}} \wedge z) \vee (x \wedge \overline{y} \wedge \overline{z}) = z,$$

hence

$$x \wedge \overline{y} \leq_{*_a} z.$$

Similarly, $\overline{x} \wedge y \leq_{*_a} z$. ■

**Lemma 2** *If elements $x, y \in L$ have the supremum in $(L, \leq_{*_a})$, then the supremum is equal to $x *_a y$.*

**Proof.** Suppose that $x$ and $y$ from $L$ have a supremum $p$ under $\leq_{*_a}$. We prove that the supremum is in the congruence relation $\phi_a$ (induced by the homomorphism $m_a$) with $x *_a y$.

Thus, $x \leq_{*_a} p$ and $y \leq_{*_a} p$. By (3), $x \leq_{*_a} x *_a y$ and $y \leq_{*_a} x *_a y$, and thus

$$p \leq_{*_a} x *_a y, \tag{7}$$

since $x *_a y$ is another lower bound of $x$ and $y$.

By Proposition 1 (vii), $\overline{p} \leq \overline{x}$, $\overline{p} \leq \overline{y}$ and $\overline{x *_a y} \leq \overline{p}$, and thus $\overline{x *_a y} \leq \overline{p} \leq \overline{x} \wedge \overline{y}$. By Proposition 1 (v),

$$\overline{p} = \overline{x} \wedge \overline{y} = \overline{x \wedge y} = \overline{x *_a y}. \tag{8}$$

Hence, $p$, $x \wedge y$ i $x *_a y$ are in same congruence class of the relation $\phi_a$. By Proposition 1$(iv)$), and by (7), $p \leq x *_a y$.

By Lemma 1, $x \wedge \overline{p} \leq p$ and $y \wedge \overline{p} \leq p$. Therefore, using (8),

$$p \leq x *_a y = (\overline{x} \wedge y) \vee (x \wedge \overline{y}) = (\overline{x} \wedge \overline{y} \wedge y) \vee (x \wedge \overline{x} \wedge \overline{y}) = (\overline{p} \wedge y) \vee (x \wedge \overline{p}) \leq p.$$

Hence, $x *_a y$ is a supremum for $x, y \in L$ under the ordering $\leq_{*_a}$ ■

By $C_{\overline{x}}$ we denote the class of the congruence $\phi_a$ (induced by homomorphism $m_a$) containing $x \in L$, and by $L/\phi_a$ the corresponding quotient set under the $\phi_a$.

**Lemma 3** *The unique minimal upper bound of elements $x, y \in L$ in $(L, \leq_{*_a})$, belonging to the class $C_{\overline{x \wedge y}}$ of the congruence $\phi_a$ is $x *_a y$.*

**Proof.** By (3), $x *_a y$ is an upper bound of elements $x, y$ in $(L, \leq_{*_a})$.

By Proposition 1 (v), $x *_a y$ belongs to the class of the congruence $\phi_a$ having the top element $\overline{x \wedge y}$.

Let $p$ be an upper bound: $x \leq_{*_a} p$, $y \leq_{*_a} p$ and let $p$ belong to the class $C_{\overline{x \wedge y}}$ of the congruence $\phi_a$. We prove that $x *_a y \leq p$. By Lemma 1, $x \wedge \overline{p} \leq p$ and $y \wedge \overline{p} \leq p$. The desired property follows similarly as in the proof of Lemma 2.

Hence, $x *_a y$ is the least of all upper bounds belonging to the class $C_{\overline{x \wedge y}}$ of the congruence $\phi_a$. ■

The statements from the two previous lemmas say that for each $x, y \in (L, \leq_{*_a})$ there is a minimal upper bound in $C_{\overline{x \wedge y}}$. Nevertheless, $x$ and $y$ may have another minimal upper bound $z$ in the class $C_{\overline{z}} \in L/\phi_a$, where $\overline{x \wedge y} \leq_{*_a} \overline{z}$. In this case, element $z$ should be incomparable with $x *_a y$. Obviously, $(L, \leq_{*_a})$ is not a lattice in this case.

**Lemma 4** *Let $(L, \leq_{*_a})$ be the poset defined by* (4) *and $x, y \in L$ such that $\overline{x} \neq \overline{y}$. Then, $x \leq_{*_a} x \wedge \overline{y}$ and $x \wedge \overline{y} \in C_{\overline{x \wedge y}}$.*

**Proof.** $x *_a (x \wedge \overline{y}) = (\overline{x} \wedge (x \wedge \overline{y})) \vee (x \wedge (\overline{x \wedge \overline{y}})) = (x \wedge \overline{y}) \vee (x \wedge \overline{x} \wedge \overline{y}) = x \wedge \overline{y}$. The second part of the lemma is straightforward. ■

Let $\mathcal{L}$ be an algebraic lattice and let $(L, \leq_{*_a})$ be the poset defined by (4). Further, let $C_{\overline{y}}$ be the class of the congruence $\phi_a$ containing $y$. For $x, y \in L$, we define the following set:

$$Q_x^y = \{z \mid z \in C_{\overline{y}}, y \leq_{*_a} x \text{ and } x \leq y\}. \tag{9}$$

It is easy to prove that if $\overline{x} = \overline{y}$ then $Q_x^y = \{x\}$. If $\overline{x} \neq \overline{y}$, then $Q_x^y$ can be empty.

Recall that an element $a$ is a lattice $L$ is **cancellable** if for every $x, y \in L$,

$$\text{from } \ a \wedge x = a \wedge y \ \text{ and } \ a \vee x = a \vee y \ \text{ it follows that } \ x = y.$$

**Lemma 5** *Codistributive element $a \in L$ is cancellable in $(L, \leq)$ if and only if for every $x, y \in L$, such that $\overline{x} \leq \overline{y}$, the set $Q_x^y$ defined by (9) is nonempty.*

**Proof.** Suppose that $a \in L$ is not cancellable. Then, there are $x, y \in L$ such that $a \vee x = a \vee y$ and $a \wedge x = a \wedge y$ and $x \neq y$. Moreover, without loss of generality we can suppose that $x$ and $y$ are comparable, because $x$ (or $y$) and $x \vee y$ satisfy the same property as $x$ and $y$. Suppose that $x < y$. We prove that $Q_x^a$ is the empty set. Take $z \in C_{\overline{a}}$, such that $x \leq z$. By $a \vee x = a \vee y$ and $a \vee x \leq z$, we obtain $a \vee y \leq z$ and $y \leq z$. Then,

$z *_a x = (\overline{z} \wedge x) \vee (z \wedge \overline{x}) = x \vee (z \wedge \overline{x}) = z \wedge \overline{x} \geq y > x.$

Therefore, $\neg(z \leq_{*_a} x)$, and $Q_x^a = \emptyset$.

On the other hand, suppose that for some $x, y \in L$, we have that $\overline{x} \leq \overline{y}$, and $Q_x^y = \emptyset$. Denote $y \wedge a$ by $b$. By $Q_x^y = Q_x^z$, for all $\overline{z} = \overline{y}$, we have that $Q_x^{b \vee x} = \emptyset$. By $x \leq b \vee x$, we have that $\neq (b \vee x \leq_{*_a} x)$, i.e., $x \vee ((b \vee x) \wedge \overline{x}) \neq x$. Thus $\neg((b \vee x) \wedge \overline{x} \leq x)$. Hence, $(b \vee x) \wedge \overline{x} \neq x$.

Moreover, $(b \vee x) \wedge \overline{x} \wedge a = ((b \wedge a) \vee (x \wedge a)) \wedge \overline{x} \wedge a = x \wedge a$.

Finally, $((b \vee x) \wedge \overline{x}) \vee a = x \vee a$.

Hence, $a$ is not cancellable. ∎

**Lemma 6** *Let $(L, \leq_{*_a})$ be the poset defined by* (4) *and $x, y \in L$. If the set $Q_x^y$ defined by (9) is nonempty, then every element from $Q_x^y$ is below a maximal element under the ordering $\leq_{*_a}$.*

**Proof.** Take a chain of elements from $Q_x^y$ :

$$\{y_i \mid i \in I\}, \text{ where } y_i \in C_{\overline{y}}, y_i \leq_{*_a} x \text{ and } x \leq y_i.$$

Denote by

$$s = \bigvee_{i \in I} y_i.$$

The supremum $s$ belongs to $C_{\overline{y}}$, because $(\bigvee_{i \in I} y_i) \wedge a = \bigvee_{i \in I}(y_i \wedge a)$ ($a$ is an infinitely distributive element in $(L, \leq)$). We show that $s \in Q_x^y$. Indeed, from the definition of the set $Q_x^y$ we have that:

$$x \leq s = \bigvee_{i \in I} y_i. \tag{10}$$

Further, from $y_i \leq_{*_a} x$, by Lemma 1 it follows that $y_i \wedge \overline{x} \leq x$, for every $i \in I$. Now, we have that $\bigvee_{i \in I}(y_i \wedge \overline{x}) \leq x$. We use a well known fact that for every $x \in L$, where $\mathcal{L}$ is an algebraic lattice and $D \subseteq L$ a directed set,

$$x \wedge \bigvee D = \bigvee_{d \in D} (x \wedge d) \text{ (see e.g. [1]).}$$

Therefore, since $\{y_i \mid i \in I\}$ is chain,

$$(\bigvee_{i \in I} y_i) \wedge \overline{x} = \bigvee_{i \in I} (y_i \wedge \overline{x}) \leq x. \tag{11}$$

Thus, by (11) we have that

$$s *_a x = (\overline{s} \wedge x) \vee ((\bigvee_{i \in I} y_i) \wedge \overline{x}) = x \vee ((\bigvee_{i \in I} y_i) \wedge \overline{x}) = x. \tag{12}$$

From (10) and (12), it follows that $s \in Q_x^y$. By Zorn lemma, if $Q_x^y$ is nonempty, then every element from $Q_x^y$ is below a maximal element under the ordering $\leq_{*_a}$ belonging to the same set. ■

**Theorem 2** *If $(L, \leq_{*_a})$ is a lattice then for all $x, y \in L$ the set $Q_x^y$ defined by (9) is empty or it has the top element under the $\leq_{*_a}$.*

**Proof.** Suppose that there are elements $x, y \in L$, $(\overline{x} \neq \overline{y})$, such that the set $Q_x^y \neq \emptyset$ does not have the top element. By Lemma 6, there are at least two maximal elements under the ordering $\leq_{*_a}$ in $Q_x^y$, say $z$ and $t$. By the definition of $Q_x^y$,
$z, t \in C_{\overline{y}}$, $z \leq_{*_a} x$, $x \leq z$, $t \leq_{*_a} x$ and $x \leq t$.

By Proposition 1 $(iv)$ $C_{\overline{y}}$ is a lattice and

$$z *_a t = z \vee t = u, \text{ where } u \in C_{\overline{y}},$$

and $u \notin Q_x^y$, by the assumption that $z$ and $t$ are maximal elements in $Q_x^y$.

By Lemma 2, if elements $z$ and $t$ have the supremum, then it is $u$. On the other hand, we proved that $x$ is an upper bound for $z$ and $t$ in $(L, \leq_{a_*})$.

Suppose that $u \leq_{*_a} x$. By $x \leq z$ and $x \leq t$, we have that $x \leq z \vee t = u$, and $u \in Q_x^y$, contrary to assumption.

Hence, $z$ and $t$ do not have the supremum in $(L, \leq_{*_a})$, and this poset is not a lattice. ■

The following example illustrates the fact that the conditions of the theorem are not sufficient for a poset $(L, \leq_{*_a})$ to be a lattice.

**Example 2** Lattice $(L, \leq)$ is given in Figure 2 a) and the corresponding poset $(L, \leq_{*_a})$ in Figure 2 b). For every $x, y \in L$, if $Q_x^y$ is nonempty, then it possesses the top element. On the other hand, $(L, \leq_{*_a})$ is not a lattice.

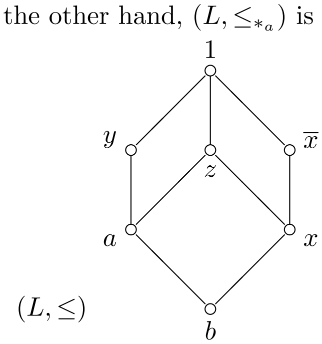


Figure 2 a)

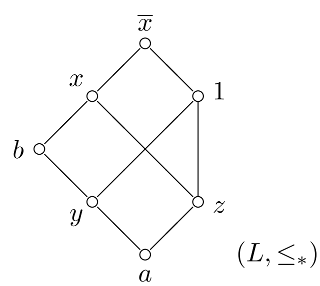


Figure 2 b)

# 3 Applications in universal algebra

In this section we give an application of the obtained results to the weak congruence lattice $(Cw\mathcal{A}, \subseteq)$ of an algebra $\mathcal{A} = (A, F)$. This lattice is algebraic, the diagonal relation $\Delta$ is always a codistributive element. The classes of the congruence $\phi_\Delta$, induced by the homomorphism $m_\Delta : \rho \longmapsto \rho \wedge \Delta$ $(\rho \in Cw\mathcal{A})$, always have top elements which are squares of subalgebras. Thus, all requirements from the previous section are fulfilled and the operation $*_\Delta$ could be introduced (see also [5]). Such an operation on the weak congruence lattice is a graphical composition (see [8]).

Let $\rho, \theta$ be weak congruences, $\rho \in Con\mathcal{B}, \theta \in Con\mathcal{C}$, for $\mathcal{B}, \mathcal{C} \in Sub\mathcal{A}$. Then,

$$\rho *_\Delta \theta = (B^2 \wedge \theta) \vee (\rho \wedge C^2) \text{ and } \qquad \emptyset *_\Delta \theta = \emptyset.$$

By the operation $*_\Delta$, as in the previous section we introduce a new ordering relation on $Cw\mathcal{A}$.

$(Cw\mathcal{A}, \leq_{*_\Delta})$ is a poset of weak congruences, where the relation $\leq_{*_\Delta}$ is defined by the operation $*_\Delta$, as follows:

$$\rho \leq_{*_\Delta} \theta \text{ if and only if } \rho *_\Delta \theta = \theta.$$

Moreover, for $\alpha \in Con\mathcal{B}$ and $\mathcal{B}, \mathcal{C} \in Sub\mathcal{A}$, we define $Q_\alpha^\mathcal{C}$ similarly to (9):

$$Q_\alpha^\mathcal{C} = \{\gamma \mid \gamma \in Con\mathcal{C}, \gamma \leq_{*_\Delta} \alpha \text{ and } \alpha \leq \gamma\}. \tag{13}$$

The propositions given in the sequel are direct consequences of the corresponding statements from the previous section.

**Theorem 3** *Let $\mathcal{A}$ be an algebra. If the weak congruence lattice $Cw\mathcal{A}$ is finite distributive lattice, then the poset $(Cw\mathcal{A}, \leq_{*_\Delta})$ is a lattice.*

**Theorem 4** *If poset $(Cw\mathcal{A}, \leq_{*_\Delta})$ is a lattice then for every $\rho \in Con\mathcal{C}$, $\mathcal{C} \in Sub\mathcal{B}$ the set $Q_\alpha^\beta$ is empty or it has the top element.*

In the sequel, we introduce propositions proved in [5], concerning the Congruence Extension Property. Recall that an algebra $\mathcal{A}$ has the Congruence Extension Property (CEP) if for every congruence $\rho$ on the subalgebra $\mathcal{B}$ of $\mathcal{A}$, there is a congruence $\theta$ on $\mathcal{A}$, such that $\theta \cap B^2 = \rho$.

**Proposition 2** *[5] A weak congruence $\theta \in Cw\mathcal{A}$ is an extension of $\rho \in Con\mathcal{B}$ if and only if $\rho \subseteq \theta$ and $\theta \leq_{*_\Delta} \rho$.*

**Proposition 3** *[5] An algebra $\mathcal{A}$ has the Congruence Extension Property (CEP) if and only if for every $\rho \in Cw\mathcal{A}$,*

$$(\rho *_\Delta A^2) \wedge (\rho \vee \Delta) = \rho.$$

**Theorem 5** *[5] An algebra $\mathcal{A}$ has the CEP if and only if for every $\rho \in Con\mathcal{B}$, $\mathcal{B} \in Sub\mathcal{A}$ there exists $\theta \in Con\mathcal{A}$, $\theta \leq_{*_\Delta} \rho$, such that from $\tau \in Con\mathcal{B}$ and $\tau \in [\theta, \rho]_{*_\Delta}$, it follows that $\rho = \tau$.*

As a direct consequence of Lemma 5, taking into account the fact that an algebra has the CEP if and only if its diagonal relation is a cancellable element in $Cw\mathcal{A}$ (see e.g. [2] or [11]), we obtain the following statement.

**Theorem 6** *An algebra $\mathcal{A}$ satisfies the CEP if and only if for all $\alpha \in Con\mathcal{B}$, $\mathcal{B}, \mathcal{C} \in Sub\mathcal{A}$, such that $B \subset C$, every set $Q_\alpha^C$ is nonempty.* □

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