FILOMAT (Niš) **18** (2004), 15–26

BOUNDING THE ČEBYŠEV FUNCTIONAL FOR SEQUENCES OF VECTORS IN NORMED LINEAR SPACES

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Abstract. Some new bounds for Čebyšev functional for sequences of vectors in normed linear spaces are found.

1. Introduction

Consider the Čebyšev functional defined for $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $\mathbf{x} = (x_1, ..., x_n) \in X^n$, where X is a linear space over the real or complex number field $\mathbb{K}$:

$$T_n(\mathbf{p}; \alpha, \mathbf{x}) := P_n \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \cdot \sum_{i=1}^{n} p_i x_i, \tag{1.1}$$

where $P_n := \sum_{i=1}^{n} p_i$.

The following Grüss type inequalities for sequences in normed linear spaces hold.

Theorem 1. *Let $(X, \|.\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{K}^n, \mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^{n} p_i = 1$ and $\mathbf{x} = (x_1, ..., x_n) \in X^n$. Then one has the inequalities*

$$\|T_n(\mathbf{p}; \alpha, \mathbf{x})\| \le \begin{cases} \left[\sum\limits_{i=1}^{n} i^2 p_i - \left(\sum\limits_{i=1}^{n} i p_i\right)^2\right] \max\limits_{1\le j\le n-1} |\Delta\alpha_j| \max\limits_{1\le j\le n-1} \|\Delta x_j\|, [1]; \\ \\ \frac{1}{2}\sum\limits_{i=1}^{n} p_i(1-p_i) \sum\limits_{j=1}^{n-1} |\Delta\alpha_j| \sum\limits_{j=1}^{n-1} \|\Delta x_j\|, [3]; \\ \\ \sum\limits_{1\le i<j\le n} p_i p_j (j-i) \left(\sum\limits_{j=1}^{n-1} |\Delta\alpha_j|^p\right)^{1/p} \left(\sum\limits_{j=1}^{n-1} \|\Delta x_j\|^q\right)^{1/q}, \\ \\ p>1, \frac{1}{p}+\frac{1}{q}=1, [2]. \end{cases} \tag{1.2}$$

2000 *Mathematics Subject Classification.* Primary 26D15; Secondary 26D10.
Key words and phrases. Čebyšev functional, Grüss Type Inequalities .
[1]Received: May 20, 2003

The constant 1 *in the first branch,* $\frac{1}{2}$ *in the second branch and* 1 *in the third branch are best possible in the sense that they cannot be replaced by smaller constants.*

The following particular inequalities for unweighted means hold as well, where $T_n(\alpha, \mathbf{x})$ is defined as follows:

$$T_n(\alpha, \mathbf{x}) := \frac{1}{n}\sum_{i=1}^{n} \alpha_i x_i - \frac{1}{n}\sum_{i=1}^{n} a_i \cdot \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Corollary 1. *With the assumptions of Theorem 1 for* X, α *and* $\mathbf{x}$*, we have*

$$\|T_n(\alpha, \mathbf{x})\| \leq \begin{cases} \frac{1}{12}\left(n^2-1\right)\max_{1\leq j\leq n-1}|\Delta\alpha_j| \max_{1\leq j\leq n-1}\|\Delta x_j\|, [1]; \\ \frac{1}{2}\cdot\left(1-\frac{1}{n}\right)\sum_{j=1}^{n-1}|\Delta\alpha_j|\sum_{j=1}^{n-1}\|\Delta x_j\|, [3]; \\ \frac{1}{6}\frac{n^2-1}{n}\left(\sum_{j=1}^{n-1}|\Delta\alpha_j|^p\right)^{1/p}\left(\sum_{j=1}^{n-1}\|\Delta x_j\|^q\right)^{1/q}, \\ p>1, \frac{1}{p}+\frac{1}{q}=1, [2]. \end{cases} \tag{1.3}$$

Here the constants $\frac{1}{12}, \frac{1}{2}$ *and* $\frac{1}{6}$ *are best possible in the sense that they cannot be replaced by smaller constants.*

For applications to estimate the p-moments of guessing mappings, see [1]. For applications in approximating the discrete Fourier transform, the discrete Mellin transform as well as some applications for polynomials and Lipschitzian mappings, see [2] and [3].

For classical results related the Čebyšev functional, see [4], [5], [6], [7], [8] and [10]. For more recent results, see [12], [13], [14], [15], [9] and [11].

2. The Identities

The first result is embodied in the following

Theorem 2. *Let* $\mathbf{p} = (p_1, ..., p_n), \mathbf{a} = (a_1, ..., a_n)$ *be* n*-tuples of real or complex numbers and* $\mathbf{x} = (x_1, ..., x_n)$ *an* n*-tuple of vectors in the linear space* X*. If we define*

$$P_i := \sum_{k=1}^{i} p_k, \bar{P}_i := P_n - P_i, i \in \{1, ..., n-1\},$$

$$A_i(\mathbf{p}) := \sum_{k=1}^{i} p_k a_k, \bar{A}_i(\mathbf{p}) := A_n(\mathbf{p}) - A_i(\mathbf{p}), i \in \{1, ..., n-1\},$$

then we have the identity

$$T_n(\mathbf{p};\mathbf{a},\mathbf{x}) = \sum_{i=1}^{n-1} \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \cdot \Delta x_i \tag{2.1}$$

$$= P_n \sum_{i=1}^{n-1} P_i \left(\frac{A_n(\mathbf{p})}{P_n} - \frac{A_i(\mathbf{p})}{P_i} \right) \cdot \Delta x_i \; (\text{if } P_i \neq 0, i \in \{1,...,n\})$$

$$= \sum_{i=1}^{n-1} P_i \bar{P}_i \left(\frac{\bar{A}_i(\mathbf{p})}{\bar{P}_i} - \frac{A_i(\mathbf{p})}{P_i} \right) \cdot \Delta x_i \left(\text{if } P_i, \bar{P}_i \neq 0, i \in \{1,...,n-1\} \right),$$

where $\Delta x_i := x_{i+1} - x_i \; (i \in \{1,...,n-1\})$ *is the forward difference.*

Proof. We use the following well known summation by parts formula

$$\sum_{l=p}^{q-1} d_l \Delta v_l = d_l v_l|_p^q - \sum_{l=p}^{q-1} v_{l+1} \Delta d_l, \tag{2.2}$$

where d_l are real or complex numbers, and v_l are vectors in a linear space, and p, q are natural numbers, $q > p$.

If we choose in (2.2), $p = 1, q = n, d_i = P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})$ and $v_i = x_i$ $(i \in \{1,...,n-1\})$, then we get

$$\begin{aligned}
& \sum_{i=1}^{n-1} (P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})) \cdot \Delta x_i \\
= \; & [P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})] \cdot x_i|_1^n - \sum_{i=1}^{n-1} \Delta (P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})) \cdot x_{i+1} \\
= \; & [P_n A_n(\mathbf{p}) - P_n A_n(\mathbf{p})] \cdot x_n - [P_1 A_n(\mathbf{p}) - P_n A_1(\mathbf{p})] \cdot x_1 \\
& - \sum_{i=1}^{n-1} [P_{i+1} A_n(\mathbf{p}) - P_n A_{i+1}(\mathbf{p}) - P_i A_n(\mathbf{p}) + P_n A_i(\mathbf{p})] \cdot x_{i+1} \\
= \; & P_n p_1 a_1 x_1 - p_1 A_n(\mathbf{p}) x_1 - \sum_{i=1}^{n-1} (p_{i+1} A_n(\mathbf{p}) - P_n p_{i+1} a_{i+1}) \cdot x_{i+1} \\
= \; & P_n p_1 a_1 x_1 - p_1 A_n(\mathbf{p}) x_1 - A_n(\mathbf{p}) \sum_{i=1}^{n-1} p_{i+1} x_{i+1} + P_n \sum_{i=1}^{n-1} p_{i+1} a_{i+1} x_{i+1} \\
= \; & P_n \sum_{i=1}^{n} p_i a_i x_i - \sum_{i=1}^{n} p_i a_i \cdot \sum_{i=1}^{n} p_i x_i \\
= \; & T_n(\mathbf{p};\mathbf{a},\mathbf{x}),
\end{aligned}$$

which produces the first identity in (2.1).

The second and the third identities are obvious and we omit the details. □

Before we prove the second result, we need the following lemma providing an identity that is interesting in itself.

Lemma 1. *Let* $\mathbf{p} = (p_1, ..., p_n)$ *and* $\mathbf{a} = (a_1, ..., a_n)$ *be* n*-tuples of real or complex numbers. Then we have the equality*

$$\det\begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} = \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j, \tag{2.3}$$

for each $i \in \{1, ..., n-1\}$.

Proof. Define, for $i \in \{1, ..., n-1\}$,

$$K(i) := \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j.$$

We have

$$\begin{aligned} K(i) &= \sum_{j=1}^{i} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j + \sum_{j=i+1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j \\ &= \sum_{j=1}^{i} P_j \bar{P}_i \cdot \Delta a_j + \sum_{j=i+1}^{n-1} P_i \bar{P}_j \cdot \Delta a_j \\ &= \bar{P}_i \sum_{j=1}^{i} P_j \cdot \Delta a_j + P_i \sum_{j=i+1}^{n-1} \bar{P}_j \cdot \Delta a_j. \end{aligned} \tag{2.4}$$

Using the summation by parts formula, we have

$$\begin{aligned} \sum_{j=1}^{i} P_j \cdot \Delta a_j &= P_j \cdot a_j\Big|_1^{i+1} - \sum_{j=1}^{i} (P_{j+1} - P_j) \cdot a_{j+1} \\ &= P_{i+1} a_{i+1} - p_1 a_1 - \sum_{j=1}^{i} p_{j+1} \cdot a_{j+1} \\ &= P_{i+1} a_{i+1} - \sum_{j=1}^{i+1} p_j \cdot a_j \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \sum_{j=i+1}^{n-1} \bar{P}_j \cdot \Delta a_j &= \bar{P}_j \cdot a_j\Big|_{i+1}^{n} - \sum_{j=i+1}^{n-1} \left(\bar{P}_{j+1} - \bar{P}_j\right) \cdot a_{j+1} \\ &= \bar{P}_n a_n - \bar{P}_{i+1} a_{i+1} - \sum_{j=i+1}^{n-1} (P_n - P_{j+1} - P_n + P_j) \cdot a_{j+1} \\ &= -\bar{P}_{i+1} a_{i+1} + \sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1}. \end{aligned} \tag{2.6}$$

Using (2.5) and (2.6) we have

$$
\begin{aligned}
K(i) &= \bar{P}_i\left(P_{i+1}a_{i+1}-\sum_{j=1}^{i+1}p_j\cdot a_j\right)+P_i\left(\sum_{j=i+1}^{n-1}p_{j+1}\cdot a_{j+1}-\bar{P}_{i+1}a_{i+1}\right)\\
&= \bar{P}_iP_{i+1}a_{i+1}-P_i\bar{P}_{i+1}a_{i+1}-\bar{P}_i\sum_{j=1}^{i+1}p_j\cdot a_j+P_i\sum_{j=i+1}^{n-1}p_{j+1}\cdot a_{j+1}\\
&= \left[(P_n-P_i)P_{i+1}-P_i(P_n-P_{i+1})\right]a_{i+1}\\
&\quad +P_i\sum_{j=i+1}^{n-1}p_{j+1}\cdot a_{j+1}-\bar{P}_i\sum_{j=1}^{i+1}p_j\cdot a_j
\end{aligned}
$$

$$
\begin{aligned}
&= P_np_{i+1}a_{i+1}+P_i\sum_{j=i+1}^{n-1}p_{j+1}\cdot a_{j+1}-\bar{P}_i\sum_{j=1}^{i+1}p_j\cdot a_j\\
&= \left(P_i+\bar{P}_i\right)p_{i+1}a_{i+1}+P_i\sum_{j=i+1}^{n-1}p_{j+1}\cdot a_{j+1}-\bar{P}_i\sum_{j=1}^{i+1}p_j\cdot a_j\\
&= P_i\sum_{j=i+1}^{n-1}p_j\cdot a_j-\bar{P}_i\sum_{j=1}^{i}p_j\cdot a_j=P_i\bar{A}_i(\mathbf{p})-\bar{P}_iA_i(\mathbf{p})\\
&= \det\begin{pmatrix} P_i & P_n\\ A_i(\mathbf{p}) & A_n(\mathbf{p})\end{pmatrix};
\end{aligned}
$$

and the identity is proved. □

We are able now to state and prove the second identity for the Čebyšev functional.

Theorem 3. *With the assumptions of Theorem 2, we have the equality*

$$
T_n(\mathbf{p};\mathbf{a},\mathbf{x})=\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}\cdot\Delta a_j\cdot\Delta x_i. \tag{2.7}
$$

The proof is obvious by Theorem 2 and Lemma 1.

Remark 1. *The identity* (2.7)*, for n-tuples of real numbers, was stated without a proof in paper* [12]*. It also may be found for the same sequences in* [9, p. 281]*, again without a proof.*

3. **Some New Inequalities**

The following result holds

Theorem 4. *Let* $(X,\|.\|)$ *be a normed linear space over the real or complex number field* $\mathbb{K}$*,* $\mathbf{a}=(a_1,...,a_n)\in\mathbb{K}^n,\mathbf{p}=(p_1,...,p_n)\in\mathbb{R}^n$ *and*

$\mathbf{x} = (x_1, ..., x_n) \in X^n$. *Then one has the inequalities*

$$\|T_n(\mathbf{p};\mathbf{a},\mathbf{x})\| \leq \begin{cases} \max_{1\leq i\leq n-1}\left|\det\begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p})\end{pmatrix}\right| \cdot \sum_{j=1}^{n-1}\|\Delta x_j\|; \\ \left(\sum_{i=1}^{n-1}\left|\det\begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p})\end{pmatrix}\right|^q\right)^{1/q} \cdot \left(\sum_{j=1}^{n-1}\|\Delta x_j\|^p\right)^{1/p} \\ \text{for } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \sum_{i=1}^{n-1}\left|\det\begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p})\end{pmatrix}\right| \cdot \max_{1\leq j\leq n-1}\|\Delta x_j\|. \end{cases} \tag{3.1}$$

All the inequalities in (3.1) *are sharp in the sense that the constants* 1 *cannot be replaced by smaller constants.*

Proof. Using the first identity in (2.1), we have

$$\|T_n(\mathbf{p};\mathbf{a},\mathbf{x})\| \leq \sum_{i=1}^{n}\left|\det\begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p})\end{pmatrix}\right|\|\Delta x_i\|.$$

Using Hölder's inequality, we deduce the desired result (3.1).

Let us prove, for instance, that the constant 1 in the second inequality is best possible.

Assume, for $C > 0$, we have that

$$\|T_n(\mathbf{p};\mathbf{a},\mathbf{x})\| \leq C\left(\sum_{i=1}^{n-1}\left|\det\begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p})\end{pmatrix}\right|^q\right)^{1/q}\left(\sum_{j=1}^{n-1}\|\Delta x_j\|^p\right)^{1/p} \tag{3.2}$$

for $p > 1, \frac{1}{p} + \frac{1}{q} = 1, n \geq 2$.

If we choose $n = 2$, then we get

$$T_2(\mathbf{p};\mathbf{a},\mathbf{x}) = p_1p_2(a_2 - a_1)(x_2 - x_1).$$

Also, for $n = 2$,

$$\left(\sum_{i=1}^{n-1}\left|\det\begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p})\end{pmatrix}\right|^q\right)^{1/q} = |p_1p_2||a_2 - a_1|$$

and

$$\left(\sum_{j=1}^{n-1}\|\Delta x_j\|^p\right)^{1/p} = \|x_2 - x_1\|.$$

Then by (3.2), holding for $n = 2, p_1, p_2 > 0, a_1 \neq a_2, x_2 \neq x_1$, we deduce $C \geq 1$, proving that 1 is the best possible constant in that inequality. $\square$

The following corollary for the uniform distribution of the probability **p** holds.

Corollary 2. *With the assumptions of Theorem 4 for* **a** *and* **x**, *we have the inequalities*

$$\|T_n(\mathbf{a},\mathbf{x})\| \leq \frac{1}{n^2}\times\begin{cases}\max_{1\leq i\leq n-1}\left|\det\begin{pmatrix} i & n \\ \sum_{k=1}^{i} a_k & \sum_{k=1}^{n} a_k\end{pmatrix}\right|\cdot\sum_{j=1}^{n-1}\|\Delta x_j\|; \\ \left(\sum_{i=1}^{n-1}\left|\det\begin{pmatrix} i & n \\ \sum_{k=1}^{i} a_k & \sum_{k=1}^{n} a_k\end{pmatrix}\right|^q\right)^{1/q}\cdot\left(\sum_{j=1}^{n-1}\|\Delta x_j\|^p\right)^{1/p} \\ \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum_{i=1}^{n-1}\left|\det\begin{pmatrix} i & n \\ \sum_{k=1}^{i} a_k & \sum_{k=1}^{n} a_k\end{pmatrix}\right|\cdot\max_{1\leq j\leq n-1}\|\Delta x_j\|.\end{cases}$$

The following result may be stated as well.

Theorem 5. *With the assumptions of Theorem 4 and if* $P_i\neq 0\,(i=1,...,n)$, *we have the inequalities*

$$\|T_n(\mathbf{p};\mathbf{a},\mathbf{x})\| \leq |P_n|\times\begin{cases}\max_{1\leq i\leq n-1}\left|\frac{A_n(\mathbf{p})}{P_n}-\frac{A_i(\mathbf{p})}{Pi}\right|\cdot\sum_{i=1}^{n-1}|P_i|\,\|\Delta x_i\|; \\ \left(\sum_{i=1}^{n-1}|P_i|\left|\frac{A_n(\mathbf{p})}{P_n}-\frac{A_i(\mathbf{p})}{Pi}\right|^q\right)^{1/q}\cdot\left(\sum_{i=1}^{n-1}|P_i|\,\|\Delta x_i\|\right)^{1/p} \\ \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum_{i=1}^{n-1}|P_i|\left|\frac{A_n(\mathbf{p})}{P_n}-\frac{A_i(\mathbf{p})}{Pi}\right|\cdot\max_{1\leq i\leq n-1}\|\Delta x_i\|.\end{cases} \tag{3.3}$$

All the inequalities in (3.3) *are sharp in the sense that the constant* 1 *cannot be replaced by a smaller constant.*

Proof. Follows by the second identity in (2.1) and taking into account that

$$\|T_n(\mathbf{p};\mathbf{a},\mathbf{x})\| \leq |P_n|\sum_{i=1}^{n-1}\left|\frac{A_n(\mathbf{p})}{P_n}-\frac{A_i(\mathbf{p})}{Pi}\right|\cdot|P_i|\,\|\Delta x_i\|.$$

Using Hölder's weighted inequality, we easily deduce (3.3).

The sharpness of the constant may be shown in a similar manner. We omit the details. □

The following corollary containing the unweighted inequalities holds.

Corollary 3. *With the above assumptions for* $\mathbf{a}$ *and* $\mathbf{x}$*, one has*

$$\|T_n(\mathbf{a},\mathbf{x})\| \leq \frac{1}{n}\times\begin{cases}\max\limits_{1\leq i\leq n-1}\left|\frac{1}{n}\sum\limits_{k=1}^{n}a_k-\frac{1}{i}\sum\limits_{k=1}^{i}a_k\right|\cdot\sum\limits_{i=1}^{n-1}i\|\Delta x_i\|; \\ \left(\sum\limits_{i=1}^{n-1}i\left|\frac{1}{n}\sum\limits_{k=1}^{n}a_k-\frac{1}{i}\sum\limits_{k=1}^{i}a_k\right|^q\right)^{1/q}\cdot\left(\sum\limits_{i=1}^{n-1}i\|\Delta x_i\|^p\right)^{1/p} \\ \text{for } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{i=1}^{n-1}i\left|\frac{1}{n}\sum\limits_{k=1}^{n}a_k-\frac{1}{i}\sum\limits_{k=1}^{i}a_k\right|\cdot\max\limits_{1\leq i\leq n-1}\|\Delta x_i\|.\end{cases} \tag{3.4}$$

The inequalities in (3.4) *are sharp in the sense mentioned above.*

Another type of inequalities may be stated if one uses the third identity in (2.1).

Theorem 6. *With the assumptions in Theorem 4 and if* $P_i, \overline{P}_i \neq 0$, $i \in \{1,...,n-1\}$, *we have the inequalities*

$$\|T_n(\mathbf{p};\mathbf{a},\mathbf{x})\| \leq \begin{cases}\max_{1\leq i\leq n-1}\left|\frac{\overline{A}_i(\mathbf{p})}{\overline{P}_i}-\frac{A_i(\mathbf{p})}{Pi}\right|\cdot\sum_{i=1}^{n-1}|P_i|\left|\overline{P}_i\right|\|\Delta x_i\|; \\ \left(\sum_{i=1}^{n-1}|P_i|\left|\overline{P}_i\right|\left|\frac{\overline{A}_i(\mathbf{p})}{\overline{P}_i}-\frac{A_i(\mathbf{p})}{Pi}\right|^q\right)^{1/q}\cdot\left(\sum_{i=1}^{n-1}|P_i|\left|\overline{P}_i\right|\|\Delta x_i\|^p\right)^{1/p} \\ \text{for } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \sum_{i=1}^{n-1}|P_i|\left|\overline{P}_i\right|\left|\frac{\overline{A}_i(\mathbf{p})}{\overline{P}_i}-\frac{A_i(\mathbf{p})}{Pi}\right|\cdot\max_{1\leq i\leq n-1}\|\Delta x_i\|.\end{cases} \tag{3.5}$$

In particular, if $p_i = \frac{1}{n}, i \in \{1, ..., n\}$*, then we have*

$$\|T_n(\mathbf{a},\mathbf{x})\| \tag{3.6}$$

$$\leq \frac{1}{n^2} \cdot \begin{cases} \max\limits_{1\leq i\leq n-1} \left| \frac{1}{n-i} \sum\limits_{k=i+1}^{n} a_k - \frac{1}{i} \sum\limits_{k=1}^{i} a_k \right| \cdot \sum\limits_{i=1}^{n-1} i(n-i) \|\Delta x_i\|; \\ \left(\sum\limits_{i=1}^{n-1} i(n-i) \left| \frac{1}{n-i} \sum\limits_{k=i+1}^{n} a_k - \frac{1}{i} \sum\limits_{k=1}^{i} a_k \right|^q \right)^{1/q} \\ \times \left(\sum\limits_{i=1}^{n-1} i(n-i) \|\Delta x_i\|^p \right)^{1/p} \quad \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum\limits_{i=1}^{n-1} i(n-i) \left| \frac{1}{n-i} \sum\limits_{k=i+1}^{n} a_k - \frac{1}{i} \sum\limits_{k=1}^{i} a_k \right| \cdot \max\limits_{1\leq i\leq n-1} \|\Delta x_i\|. \end{cases}$$

The inequalities in (3.5) *and* (3.6) *are sharp in the above mentioned sense.*

A different approach may be considered if one uses the representation in terms of double sums for the Čebyšev functional provided by the Theorem 3.

The following result holds.

Theorem 7. *With the assumptions in Theorem 4, we have the inequalities*

$$\|T_n(\mathbf{p};\mathbf{a},\mathbf{x})\| \tag{3.7}$$

$$\leq \begin{cases} \max_{1\leq i,j\leq n-1} \left\{ \left| P_{\min\{i,j\}} \right| \left| \bar{P}_{\max\{i,j\}} \right| \right\} \cdot \sum_{i=1}^{n-1} |\Delta a_i| \sum_{i=1}^{n-1} \|\Delta x_i\|; \\ \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left| P_{\min\{i,j\}} \right|^q \left| \bar{P}_{\max\{i,j\}} \right|^q \right)^{1/q} \\ \times \left(\sum_{i=1}^{n-1} |\Delta a_i|^p \right)^{1/p} \left(\sum_{i=1}^{n-1} \|\Delta x_i\|^p \right)^{1/p} \quad \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left| P_{\min\{i,j\}} \right| \left| \bar{P}_{\max\{i,j\}} \right| \\ \times \max_{1\leq i\leq n-1} |\Delta a_i| \max_{1\leq i\leq n-1} \|\Delta x_i\|. \end{cases}$$

The inequalities are sharp in the sense mentioned above.

The proof follows by the identity (2.7) on using Hölder's inequality for double sums and we omit the details.

Now, define

$$k_\infty := \max_{1\leq i,j\leq n-1} \left\{ \frac{\min\{i,j\}}{n} \left(1 - \frac{\max\{i,j\}}{n} \right) \right\}, n \geq 2.$$

Using the elementary inequality

$$ab \leq \frac{1}{4}(a+b)^2, \quad a,b \in R;$$

we deduce

$$\begin{aligned} \min\{i,j\} \cdot (n - \max\{i,j\}) &\leq \frac{1}{4}(n + \min\{i,j\} - \max\{i,j\})^2 \\ &= \frac{1}{4}(n - |i-j|)^2, 1 \leq i,j \leq n-1. \end{aligned}$$

Consequently, we observe that

$$k_\infty \leq \frac{1}{4n^2} \max_{1\leq i,j\leq n-1} \left\{(n-|i-j|)^2\right\} = \frac{1}{4}.$$

We may state now the following corollary of Theorem 7.

Corollary 4. *Let* $(X, \|.\|)$ *be a normed linear space,* $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{K}^n$ *and* $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{X}^n$. *Then we have the inequality*

$$\|T_n(\mathbf{a},\mathbf{x})\| \leq k_\infty \sum_{i=1}^{n-1} |\Delta a_i| \sum_{i=1}^{n-1} \|\Delta x_i\| \leq \frac{1}{4} \sum_{i=1}^{n-1} |\Delta a_i| \sum_{i=1}^{n-1} \|\Delta x_i\|. \tag{3.8}$$

The constant $\frac{1}{4}$ *cannot be replaced in general by a smaller constant.*

Remark 2. *The inequality* (3.8) *is better than the second inequality in Corollary 1.*

Consider now, for $q > 1$, the number

$$k_q := \frac{1}{n^2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}[\min\{i,j\} \cdot (n-\max\{i,j\})]^q\right)^{1/q}.$$

We observe, by the symmetry of the terms under the sums symbol, we have that

$$k_q = \frac{1}{n^2}\left(2\sum_{1\leq i<j\leq n-1} i^q (n-j)^q + \sum_{i=1}^{n-1} i^q (n-i)^q\right)^{1/q},$$

that may be computed exactly if $q = 2$ or another natural number.

Since, as above,

$$[\min\{i,j\} \cdot (n-\max\{i,j\})]^q \leq \frac{1}{4^q}(n-|i-j|)^{2q}$$

we deduce

$$\begin{aligned} k_q &\leq \frac{1}{4n^2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(n-|i-j|)^{2q}\right)^{1/q} \\ &\leq \frac{1}{4n^2}\left[(n-1)^2 n^{2q}\right]^{1/q} = \frac{1}{4}(n-1)^{2/q}. \end{aligned}$$

Consequently, we may state the following corollary.

Corollary 5. *With the assumption in Corollary 4, we have the inequalities*

$$\begin{aligned}\|T_n(\mathbf{a},\mathbf{x})\| &\le k_q\left(\sum_{i=1}^{n-1}|\Delta a_i|^p\right)^{1/p}\left(\sum_{i=1}^{n-1}\|\Delta x_i\|^p\right)^{1/p}\\ &\le \frac{1}{4}(n-1)^{2/q}\left(\sum_{i=1}^{n-1}|\Delta a_i|^p\right)^{1/p}\left(\sum_{i=1}^{n-1}\|\Delta x_i\|^p\right)^{1/p};\end{aligned}$$

provided $p>1, \frac{1}{p}+\frac{1}{q}=1$. *The constant* $\frac{1}{4}$ *cannot be replaced in general by a smaller constant.*

Finally, if we denote

$$k_1 := \frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left[\min\{i,j\}\cdot(n-\max\{i,j\})\right],$$

then we observe, for $\mathbf{u}=\left(\frac{1}{n},...,\frac{1}{n}\right), \mathbf{e}=(1,2,...,n)$, that

$$k_1 = T_n(\mathbf{u};\mathbf{e},\mathbf{e}) = \frac{1}{n}\sum_{i=1}^{n} i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} i\right)^2 = \frac{1}{12}\left(n^2-1\right),$$

and by Theorem 7, we deduce the inequality

$$\|T_n(\mathbf{a},\mathbf{x})\| \le \frac{1}{12}\left(n^2-1\right)\max_{1\le j\le n-1}|\Delta a_j|\max_{1\le j\le n-1}\|\Delta x_j\|.$$

Note that, the above inequality, has been discovered by a different method in [1]. The constant $\frac{1}{12}$ is best possible.

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