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DIRECT LOCAL AND GLOBAL APPROXIMATION THEOREM FOR BERNSTEIN - TYPE OPERATORS

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ABSTRACT. Direct local and global estimates are established for Bernstein - type operators using Ditzian - Totik modulus of smoothness of second order.

1. **Introduction**

In [2] it was shown, among others, that for the Bernstein operator

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x)\ f\left(\frac{k}{n}\right), \quad p_{n,k}(x) = \left(\begin{array}{c} n \\ k \end{array}\right) x^k (1-x)^{n-k}$$

the estimate

$$|B_n(f,x) - f(x)| \ \leq\ C_1 \cdot \omega_\phi^2 \left(f, n^{-1/2}\ \frac{\varphi(x)}{\phi(x)}\right) \tag{1}$$

holds true for $x \in [0,1]$, $\varphi(x) = \sqrt{x(1-x)}$ and $f \in C[0,1]$, where the Ditzian - Totik modulus of second order is given by

$$\omega_\phi^2(f,\delta) \ =\ \sup_{0<h\leq\delta}\ \sup_{x\pm h\phi(x)\in[0,1]}\ |\ f(x+h\phi(x)) - 2f(x) + f(x-h\phi(x))\ |,$$

in which $\phi : [0,1] \to \mathbf{R}$ is an admissible step - weight function of the Ditzian - Totik modulus (see [1, p. 8]) such that ϕ^2 is a concave function.

The aim of the paper is to give a direct local and global approximation theorem for Bernstein - type operators, similar to (1), replacing the condition ϕ^2 is concave function.

In what follows we suppose that $\phi : [0,1] \to \mathbf{R}$ is an admissible step - weight function (see [1, Section 1.2]) such that $\beta = \beta(0) = \beta(1) \in [0,1/2]$ and $\phi(x) \sim x^\beta$ as $x \to 0+$, $\phi(x) \sim (1-x)^\beta$ as $x \to 1-$. Then, by [1, p. 24, Theorem 3.1.2] we have that the K - functional $\bar{K}_{2,\phi}(f,\delta^2)$ and the Ditzian - Totik modulus $\omega_\phi^2(f,\delta)$ are equivalent. Here

$$\bar{K}_{2,\phi}(f,\delta^2) = \inf_{g'\in A.C._{loc}[0,1]}\ \left\{\ \|f-g\| + \delta^2\ \|\phi^2 g''\| + \delta^{2/(1-\beta)}\ \|g''\|\ \right\}$$

and $\|\cdot\|$ denotes the uniform norm on $[0,1]$.

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Furthermore, we need the following definition : the function $\psi : [0,1] \to \mathbf{R}$ is *quasi - convex* if for every $x_1, x_2 \in [0,1]$ and v between x_1 and x_2 we have $\psi(v) \le \max\{\psi(x_1), \psi(x_2)\}$.

2. The direct approximation theorem

Our main result is the following

Theorem 1. *Let $L_n : C[0,1] \to C[0,1]$ be a sequence of uniformly bounded positive linear operators such that $L_n(u-x,x) \equiv 0$ and $L_n((u-x)^2,x) \le C_2 \cdot \frac{\varphi^2(x)}{n}$, $x \in [0,1]$, $n = 1,2,\ldots$ If there exists $\lambda \in [0,1]$ with the properties*

(i) the function $\psi(x) = \frac{x^\lambda(1-x)^\lambda}{\phi^2(x)}$ is quasi - convex on $[0,1]$ and

(ii) $L_n\left(\frac{(u-x)^2}{x^\lambda(1-x)^\lambda} \cdot \left(\frac{x^\lambda(1-x)^\lambda}{\phi^2(x)} + \frac{u^\lambda(1-u)^\lambda}{\phi^2(u)}\right), x\right) \le \frac{C_3}{n} \cdot \frac{\varphi^2(x)}{\phi^2(x)}$ for $x \in \left[\frac{1}{n}, 1-\frac{1}{n}\right]$

then the pointwise approximation

$$|L_n(f,x) - f(x)| \le C\, \omega_\phi^2\left(f, n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)}\right) \tag{2}$$

holds true for $x \in [0,1]$, $n = 1,2,\ldots$ and $f \in C[0,1]$, where the constant C depends only on ϕ.

Remark 1. *$C_1, C_2, C_3, C_4, C_5, C_6$ denote absolute positive constants and $C > 0$ is an absolute constant which depends only on ϕ. The value of C may vary with each occurrence.*

To prove our theorem we need the following two lemmas :

Lemma 1. *For $x, u \in (0,1)$, $\lambda \in [0,1]$ and v between x and u we have*

$$\frac{|u-v|}{v^\lambda(1-v)^\lambda} \le \frac{|u-x|}{x^\lambda(1-x)^\lambda}.$$

Proof. If $x \le v \le u$ then $1/v^\lambda \le 1/x^\lambda$ and the function $h(v) = (u-v)/(1-v)^\lambda$ is decreasing on $[x,u]$. So $h(v) \le h(x)$. Hence $h(v)/v^\lambda \le h(x)/x^\lambda$. Analogously, if $u \le v \le x$ then $1/(1-v)^\lambda \le 1/(1-x)^\lambda$ and the function $h(v) = (v-u)/v^\lambda$ is increasing on $[u,x]$. Therefore $h(v) \le h(x)$ and $h(v)/(1-v)^\lambda \le h(x)/(1-x)^\lambda$, which was to be proved.

Lemma 2. *For $x \in [0,1/n) \cup (1-1/n,1]$ and $\alpha \in [0,1]$ we have*

$$\frac{\varphi^2(x)}{n} \le \left(\frac{\varphi^2(x)}{x^\alpha(1-x)^\alpha} \cdot \frac{1}{n}\right)^{\frac{2}{2-\alpha}}.$$

Proof. We have to prove that

$$\frac{x(1-x)}{n} \le \left(x^{1-\alpha}(1-x)^{1-\alpha} \cdot \frac{1}{n}\right)^{\frac{2}{2-\alpha}}.$$

This is equivalent with

$$\left(x(1-x)\cdot\frac{1}{n}\right)^{1-\frac{\alpha}{2}} \le x^{1-\alpha}(1-x)^{1-\alpha}\cdot\frac{1}{n}$$

or $x^{\frac{\alpha}{2}}\,(1-x)^{\frac{\alpha}{2}} \le (1/n)^{\frac{\alpha}{2}}$ or $x(1-x) \le 1/n$. But the last estimate is true for $x\in[0,1/n)\cup(1-1/n,1]$.

Proof of the theorem. Let $x\in[1/n,1-1/n]$ and $g'\in A.C._{loc}[0,1]$. In view of Taylor's expansion

$$g(u) = g(x)+(u-x)\,g'(x)+\int_x^u (u-v)\,g''(v)\,dv$$

and $L_n(u-x,x)\equiv 0$ we obtain

$$\begin{aligned}
|L_n(g,x)-g(x)| &\le L_n\left(\Big|\int_x^u (u-v)\,g''(v)\,dv\Big|,\ x\right)\\
&\le L_n\left(\Big|\int_x^u \frac{|u-v|}{\phi^2(v)}\cdot\phi^2(v)|g''(v)|\,dv\Big|,\ x\right)\\
&\le L_n\left(\Big|\int_x^u \frac{|u-v|}{\phi^2(v)}\,dv\Big|,\ x\right)\cdot\|\phi^2g''\|\\
&= L_n\left(\Big|\int_x^u \frac{|u-v|}{v^\lambda(1-v)^\lambda}\cdot\frac{v^\lambda(1-v)^\lambda}{\phi^2(v)}\,dv\Big|,\ x\right)\cdot\|\phi^2g''\| \qquad (3)
\end{aligned}$$

By (i), we have for v between x and u that $\psi(v)\le\max\{\psi(x),\psi(u)\}\le\psi(x)+\psi(u)$. Hence, in view of (3), Lemma 1 and (ii) we have $|L_n(g,x)-g(x)| \le$

$$\begin{aligned}
&\le L_n\left(\Big|\int_x^u \frac{|u-v|}{v^\lambda(1-v)^\lambda}\cdot\left(\frac{x^\lambda(1-x)^\lambda}{\phi^2(x)}+\frac{u^\lambda(1-u)^\lambda}{\phi^2(u)}\right)dv\Big|,\ x\right)\cdot\|\phi^2g''\|\\
&\le L_n\left(\frac{(u-x)^2}{x^\lambda(1-x)^\lambda}\cdot\left(\frac{x^\lambda(1-x)^\lambda}{\phi^2(x)}+\frac{u^\lambda(1-u)^\lambda}{\phi^2(u)}\right),\ x\right)\cdot\|\phi^2g''\|\\
(4)\ &\le \frac{C_3}{n}\cdot\frac{\varphi^2(x)}{\phi^2(x)}\cdot\|\phi^2g''\|
\end{aligned}$$

Furthermore, if $x\in[0,1/n)\cup(1-1/n,1]$ and $g'\in A.C._{loc}[0,1]$ then from

$$g(u) = g(x)+(u-x)\,g'(x)+\int_x^u (u-v)\,g''(v)\,dv$$

we obtain $|L_n(g,x)-g(x)| \le L_n((u-x)^2,x)\cdot\|g''\|$. Using $L_n((u-x)^2,x) \le C_2\cdot\frac{\varphi^2(x)}{n}$ and Lemma 2 for $\alpha=2\beta\in[0,1]$ we get

$$\begin{aligned}
|L_n(g,x)-g(x)| &\le C_2\cdot\frac{\varphi^2(x)}{n}\cdot\|g''\| \le C_2\cdot\left(\frac{\varphi^2(x)}{x^{2\beta}(1-x)^{2\beta}}\cdot\frac{1}{n}\right)^{\frac{2}{2-2\beta}}\cdot\|g''\|\\
&= C_2\cdot\left(\frac{\varphi^2(x)}{\phi^2(x)}\cdot\frac{\phi^2(x)}{x^{2\beta}(1-x)^{2\beta}}\cdot\frac{1}{n}\right)^{\frac{2}{2-2\beta}}\cdot\|g''\| \qquad (5)
\end{aligned}$$

But for ϕ we have $\beta = \beta(0) = \beta(1)$. So there exists $C = C(\phi) > 0$ such that

$$\frac{\phi^2(x)}{x^{2\beta}(1-x)^{2\beta}} \le C \qquad \text{for} \quad x \in \left[\, 0, \frac{1}{n} \right) \cup \left(1 - \frac{1}{n}, 1 \,\right].$$

Hence, by (5) we have

(6)

$$|L_n(g,x) - g(x)| \le C \cdot \left(\frac{\varphi^2(x)}{\phi^2(x)} \cdot \frac{1}{n} \right)^{\frac{2}{2-2\beta}} \cdot \|g''\| = C \cdot \left(n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)} \right)^{\frac{2}{1-\beta}} \cdot \|g''\|$$

In conclusion (4) and (6) imply

$|L_n(g,x) - g(x)| \le$

$$(7) \qquad \le C \cdot \left\{ \left(n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)} \right)^2 \cdot \|\phi^2 g''\| + \left(n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)} \right)^{\frac{2}{1-\beta}} \cdot \|g''\| \right\}$$

for every $x \in [0,1]$.

On the other hand, $\{L_n\}$ is a sequence of uniformly bounded operators, therefore $\|L_n f\| \le C_4 \, \|f\|$ for $n = 1, 2, \ldots$ and $f \in C[0,1]$. Hence, by (7) we obtain

$|L_n(f,x) - f(x)| \le$

$$\begin{aligned} &\le |L_n(f-g,x) - (f-g)(x)| + |L_n(g,x) - g(x)| \\ &\le (C_4 + 1) \, \|f - g\| + |L_n(g,x) - g(x)| \\ &\le C \left\{ \|f - g\| + \left(n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)} \right)^2 \cdot \|\phi^2 g''\| + \left(n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)} \right)^{\frac{2}{1-\beta}} \cdot \|g''\| \right\} \end{aligned}$$

Using the equivalence between $\bar{K}_{2,\phi}\left(f \,,\, \left(n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)} \right)^2 \right)$ and $\omega_\phi^2 \left(f \,,\, n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)} \right)$ we get the assertion of our theorem.

Remark 2. *There are functions $\phi : [0,1] \to \mathbf{R}$ such that*

$$\psi(x) = \frac{x(1-x)}{\phi^2(x)}, \qquad x \in [0,1]$$

is quasi - convex and ϕ^2 is non - concave.

Indeed, the function

$$\psi(x) = \begin{cases} 1, & \text{if} \quad x \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right] \\ -2x + \frac{3}{2}, & \text{if} \quad x \in \left(\frac{1}{4}, \frac{1}{2}\right] \\ 2x - \frac{1}{2}, & \text{if} \quad x \in \left(\frac{1}{2}, \frac{3}{4}\right) \end{cases}$$

is quasi - convex on $[0,1]$ and

$$\phi^2(x) = \begin{cases} x(1-x), & \text{if} \quad x \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right] \\ \frac{x(1-x)}{-2x+\frac{3}{2}}, & \text{if} \quad x \in \left(\frac{1}{4}, \frac{1}{2}\right] \\ \frac{x(1-x)}{2x-\frac{1}{2}}, & \text{if} \quad x \in \left(\frac{1}{2}, \frac{3}{4}\right) \end{cases}$$

is not a concave function on $[0,1]$, because

$$\frac{d^2}{dx^2}\left(\frac{x(1-x)}{3-4x}\right) \;=\; \frac{6(3-4x)}{(3-4x)^4} \;>\; 0$$

for $x \in (1/4, 1/2)$.

3. **Applications**

In this section we shall apply our theorem for some well - known positive linear operators, when $\phi^2(x) = x^\lambda(1-x)^\lambda$, $0 \le \lambda \le 1$. So we obtain estimates similar to (2), which combine the classical estimate ($\lambda = 0$) and the estimate developed by Ditzian - Totik ($\lambda = 1$).

Let us observe that for $\phi^2(x) = x^\lambda(1-x)^\lambda$ the conditions (i) and (ii) are satisfied. Indeed, in this case $\psi(x) = 1$, $x \in [0,1]$, which is obviously quasi - convex and

$$L_n\left(\tfrac{(u-x)^2}{x^\lambda(1-x)^\lambda}\cdot\left(\tfrac{x^\lambda(1-x)^\lambda}{\phi^2(x)}+\tfrac{u^\lambda(1-u)^\lambda}{\phi^2(u)}\right)\;,\;x\right) \;=$$

$$= \; \frac{2}{x^\lambda(1-x)^\lambda}\cdot L_n\left((u-x)^2, x\right)$$

$$\le \; \frac{2C_2}{n}\cdot\frac{\varphi^2(x)}{x^\lambda(1-x)^\lambda} \;=\; \frac{2C_2}{n}\cdot\frac{\varphi^2(x)}{\phi^2(x)},$$

$x \in [1/n, 1-1/n]$, respectively. Therefore we have the following results:

a) *Stancu operator* [5] is defined by

$$S_n^t(f,x) \;=\; \sum_{k=0}^{n} w_{n,k}(x,t)\; f\left(\frac{k}{n}\right),$$

where $x \in [0,1]$, $f \in C[0,1]$ and

$$w_{n,k}(x,t) \;=\; \begin{pmatrix} n \\ k \end{pmatrix} \frac{\Pi_{i=0}^{k-1}\,(x+it)\;\Pi_{j=0}^{n-k-1}\,(1-x+jt)}{\Pi_{r=0}^{n-1}\,(1+rt)},$$

t is a positive parameter which may depends only on the natural number n. We have $S_n^t(u-x,x) \equiv 0$ and

$$S_n^t((u-x)^2,x) \;=\; \frac{1+nt}{n(1+t)}\cdot x(1-x) \;\le\; \frac{C_5}{n}\cdot x(1-x)$$

if $t = t(n) = O(n^{-1})$. So

$$|S_n^t(f,x) - f(x)| \;\le\; C\;\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\;\varphi^{1-\lambda}(x)\right),$$

where $x \in [0,1]$, $f \in C[0,1]$, $t = t(n) = O(n^{-1})$, $n = 1, 2, \ldots$

b) *Goodman - Sharma operator* [4] is defined by

$U_n(f,x) \;=$

$$= \; f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + \sum_{k=1}^{n-1} p_{n,k}(x)\;\int_0^1 (n-1)\;p_{n-2,k-1}(t)\;f(t)\;dt,$$

where $x \in [0,1]$ and $f \in C[0,1]$. We have $U_n(u-x,x) \equiv 0$ and

$$U_n((u-x)^2,x) \ = \ \frac{2}{n+1} \cdot x(1-x) \ \leq \ \frac{2}{n} \cdot x(1-x).$$

Then

$$|U_n(f,x)-f(x)| \ \leq \ C\ \omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\ \varphi^{1-\lambda}(x)\right),$$

$x \in [0,1]$, $f \in C[0,1]$, $n = 1,2,\ldots$

c) *The generalized Goodman - Sharma operator* was defined in [3] as follows :

$U_n^t(f,x) \ =$

$$= \ f(0)w_{n,0}(x,t)+f(1)w_{n,n}(x,t)+\sum_{k=1}^{n-1} w_{n,k}(x,t) \int_0^1 (n-1)\, p_{n-2,k-1}(t)\, f(t)\, dt,$$

where $x \in [0,1]$, $f \in C[0,1]$ and $w_{n,k}(x,t)$ is given in a). We have $U_n^t(u-x,x) \equiv 0$ and

$$U_n^t((u-x)^2,x) \ = \ \frac{2+(n+1)t}{(n+1)(1+t)} \cdot x(1-x) \ \leq \ \frac{C_6}{n} \cdot x(1-x)$$

if $t = t(n) = O((n+1)^{-1})$. So

$$|U_n^t(f,x)-f(x)| \ \leq \ C\ \omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\ \varphi^{1-\lambda}(x)\right),$$

where $x \in [0,1]$, $f \in C[0,1]$, $t = t(n) = O((n+1)^{-1})$, $n = 1,2,\ldots$

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