A GENERAL COMMON FIXED POINT THEOREM OF MEIR AND KEELER TYPE FOR NONCONTINUOUS WEAK COMPATIBLE MAPPINGS

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Abstract

In this paper,using a combination of methods used in [1],[20] and [22] the results from [3, Theorem 1], [14,Theorem 1] and [15, Theorem 1] are improved by removing the assumption of continuity,relaxing compatibility to weak compatibility property and replacing the completeness of the space with a set of four alternative conditions for functions satisfying an implicit relation.

1 Introduction

Let S and T be two self mappings of a metric space (X,d). Jungck [3] defines S and T to be compatible if $lim\, d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $lim\, Sx_n = lim\, Tx_n = x$ for some $x \in X$. In 1993, Jungck, Murthy and Cho [6] define S and T to be compatible of type (A) if $lim\, d(TSx_n, S^2x_n) = 0$ and $lim\, d(STx_n, T^2x_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $lim\, Sx_n = lim\, Tx_n = x$ for some $x \in X$.

By [6, Ex.2.1 and Ex.2.2] it follows that the notions of compatible mappings and compatible mappings of type (A) are independent.

Recently, Pathak and Khan [17] introduced a new concept of compatible of type (B) as a generalization of compatible mappings of type (A). S and T is said to be compatible of type (B) if

$$lim\, d(STx_n, T^2x_n) \leq \frac{1}{2}[lim\, d(STx_n, St) + lim\, d(St, S^2x_n)],$$

$$lim\, d(TSx_n, S^2x_n) \leq \frac{1}{2}[lim\, d(TSx_n, Tt) + lim\, d(Tt, T^2x_n)],$$

whenever $\{x_n\}$ is a sequence in X such that $lim\, Sx_n = lim\, Tx_n = t$ for some $t \in X$.

Clearly, compatible mappings of type (A) are compatible of type (B). By [17, Ex.2.4] it follows that implication is not reversible.

In [18] the concept of compatible mappings of type (P) was introduced and compared with compatible mappings and compatible mappings of type (A). S and T are compatible of type (P) if $lim\, d(S^2x_n, T^2x_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $lim\, Sx_n = lim\, Tx_n = t$ for some $t \in X$.

Lemma 1 [4] (resp. [6],[17].[18]). Let S and T be compatible (resp. compatible of type (A), compatible of type (B), compatible of type (P)) self mappings of a metric space (X,d). If $Sx = Tx$ for some $x \in X$, then $STx = TSx$.

In 1994, Pant [11] introduced the notion of pointwise R-weakly commuting mappings. It is proved in [12] that the notion of pointwise R-weakly commuting is equivalent to commutativity in coincidence points. Jungck [5] defines S and T to be weakly compatible if $Sx = Tx$ implies $STx = TSx$. Thus S and T are weakly compatible if and only if S and T are pointwise R-weakly commuting mappings.

Remark 1. By Lemma 1 it follows that every compatible (compatible of type (A), compatible of type (B), compatible of type (P)) pair of mappings is weakly compatible .

The following example is an example of weakly compatible mappings which is not compatible (compatible of type (A), compatible of type (B), compatible of type (P)).

Let $X = [2, 20]$ with the usual metric. Define

$$Tx = \begin{cases} 2 & \text{if } x = 2 \\ 12 + x & \text{if } 2 < x \le 5; \\ x - 3 & \text{if } 5 < x \le 20 \end{cases} Sx = \begin{cases} 2 & \text{if } x \in \{2\} \cup (5, 20] \\ 8 & \text{if } \quad 2 < x \le 5 \end{cases}$$

S and T are weakly compatible since they commute and their coincidence point [12] but S and T are not compatible.

By [19] S and T are not compatible of type (A) and noncompatible of type (P). S and T are not compatible of type (B). Indeed, let us consider a decreasing sequence $\{x_n\}$ such that $lim\, x_n = 5$, then $lim\, Tx_n = 2$, $lim\, Sx_n = 2$; $lim\, STx_n = 8, lim\, T^2x_n = 14, lim\, S^2x_n = 2$. Then $lim\, d(STx_n, T^2x_n) = 6 > \frac{1}{2}[lim\, STx_n, St) + lim\, d(St, S^2x_n)] = \frac{1}{2}[6 + 0] = 3$.

2   Preliminaries

In 1969, Meir and Keeler [8] established a fixed point theorem for self mappings of a metric space (X,d) satisfying the following condition:

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

(2.1) $\varepsilon \leq d(x, y) < \varepsilon + \delta$ implies $d(fx, fy) < \varepsilon$.

There exists a vast literature which generalize the result of Meir and Keeler. In [7] Maiti and Pal proved a fixed point theorem for a self mappings f of a metric space (X,d) satisfying the following condition, which is a generalization of (2.1):

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

(2.2) $\varepsilon \leq max\{d(x, y), d(x, fx), d(y, fy)\} < \varepsilon + \delta$

implies $d(fx, fy) < \varepsilon$.

In [16] and [21], Park-Rhodes and Rao-Rao extended this result for two mappings and proved some fixed point theorems for self mappings f and g of a metric space (X,d) satisfying the following condition:

(2.3) $\varepsilon \leq max\{d(fx, fy), d(fx, gx), d(fy, gy), \frac{1}{2}[d(fx, gy)+d(fy, gx)]\} < \varepsilon + \delta$

implies $d(gx, gy) < \varepsilon$.

In 1986, Jungck [4] and Pant [9] extended these results for four mappings. It is known from Jungck [4] and Pant [10],[12]-[14] and other papers the fact that in case of theorems for four mappings A,B,S,T : $(X, d) \rightarrow (X, d)$, a condition of Meir-Keeler type didn't assure the existence of a fixed point.

The following theorem is proved in [3].

Theorem 1[3]. Let (A,S) and (B,T) be compatible mappings of a complete metric space (X,d) such that :

(i) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(ii) given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x,y in X

$\varepsilon \leq max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty); \frac{1}{2}[d(Sx, By) + d(Ax, Ty)]\} < \varepsilon + \delta$

implies $d(Ax, By) < \varepsilon$ and (iii) $d(Ax, By) < k[d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) + d(Sx, By) + d(Ax, Ty)]$

for $0 \leq k \leq \frac{1}{3}$. If one of mappings A,B,S and T is continuous then A,B,S and T have a unique fixed point.

The following theorems appear in [14], respectively [15].

Theorem 2 [14]. Let A,B,S and T as in Theorem 1 satisfying (i), (ii) and (iv) $d(Ax, By) < max\{k_1 d(Sx, Ty), k_2[d(Ax, Sx)+d(By, Ty)]/2; [d(Sx, By)+d(Ax, Ty)]/2\}$ for $k_1 \geq 0$ and $1 \leq k_2 < 2$.

If one of the mappings A,B,S and T is continuous then A,B,S and T have a unique fixed point.

Theorem 3 [15]. Let A,B,S and T as in Theorem 1 satisfying (i), (ii) and (v) $d(Ax, By) < max\{d(Sx, Ty), [d(Ax, Sx) + d(By, Ty)]/2, k[d(Sx, By) + d(Ax, Ty)]\}$ for $1 \leq k < 2$.

If one of the mappings A,B,S and T is continuous, then A,B,S and T have a unique common fixed point.

3   Implicit relations

Let $\mathcal{F}_6$ be the set of all continuous functions $F(t_1, ..., t_6) : R_+^6 \rightarrow R$ satisfying the following conditions:

$(F_1) : F(u, 0, u, 0, 0, u) \leq 0$ implies $u = 0$; $(F_2) : F(u, 0, 0, u, u, 0) \leq 0$ implies $u = 0$.

The function $F(t_1, ...t_6) : R_+^6 \rightarrow R$ satisfies the conditions (F_u) if

$$F(u, u, 0, 0, u, u) \geq 0; \forall u > 0$$

Ex.1. $F(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6)$, where
$a, b, c \geq 0$, $0 \leq b + c < 1$ and $0 \leq a + 2c \leq 1$
$(F_1) : F(u, 0, u, 0, 0, u) = u(1 - b - c) \leq 0$ implies $u = 0$.
Similarly, $F(u, 0, 0, u, u, 0) \leq 0$ implies $u = 0$.
$(F_u) : F(u, u, 0, 0, u, u) = u(1 - a - 2c) \geq 0; \forall u > 0$.

Ex.2. $F(t_1, ..., t_6) = t_1 - max\{t_2, (t_3 + t_4)/2, k(t_5 + t_6)/2\}$ where $0 \leq k < 1$
$(F_1) : F(u, 0, u, 0, 0, u) = u - max\{\frac{u}{2}, k\frac{u}{2}\} = \frac{u}{2} \leq 0$ implies $u = 0$.
Similarly, $F(u, 0, 0, u, u, 0) \leq 0$ implies $u = 0$.
$(F_u) : F(u, u, 0, 0, u, u) = u - max\{u, ku\} = 0; \forall u > 0$.

Ex.3. $F(t_1, ...t_6) = t_1 - max\{k_1 t_2, k_2(t_3 + t_4)/2, (t_5 + t_6)/2\}$ where $0 \leq k_1 < 1$, $1 \leq k_2 < 2$.
$(F_1) : F(u, 0, u, 0, 0, u) = u - max\{k_2 u/2, \frac{u}{2}\} = u(1 - \frac{k_2}{2}) \leq 0$ implies $u = 0$.
Similarly, $F(u, 0, 0, u, u, 0) \leq 0$ implies $u = 0$.
$(F_u) : F(u, u, 0, 0, u, u) = u - max\{k_1 u, u\} = 0; \forall u > 0$.

Ex.4. $F(t_1, ...t_6) = t_1 - hmax\{t_2, t_3, t_4, t_5, t_6\}$, where $0 \leq h < 1$. $(F_1) : F(u, 0, u, 0, 0, u) = u(1 - h) \leq 0$ implies $u = 0$.

Similarly, $F(u, 0, 0, u, u, 0) \le 0$ implies $u = 0$.
$(F_u) : F(u, u, 0, 0, u, u) = u(1 - h) \ge 0; \forall u > 0$.

Ex.5. $F(t_1, ...t_6) = t_1^2 - at_2^2 - t_3 t_4 - bt_5^2 - ct_6^2$, where $a, b, c \ge 0, 0 \le a + b + c < 1$.
$(F_1) : F(u, 0, u, 0, 0, u) = u^2(1 - c) \le 0$ implies $u = 0$.
$(F_2) : F(u, 0, 0, u, u, 0) = u^2(1 - b) \le 0$ implies $u = 0$.
$(F_u) : F(u, u, 0, 0, u, u) = u^2(1 - a - b - c) \ge 0; \forall u > 0$.

Ex.6. $F(t_1, ...t_6) = t_1^3 - k(t_2^3 + t_3^3 + t_4^3 + t_5^3 + t_6^3)$, where $0 \le k \le \frac{1}{3}$.
$(F_1) : F(u, 0, u, 0, 0, u) = u^2(1 - 2k) \le 0$ implies $u = 0$.
Similarly, $F(u, 0, 0, u, u, 0) \le 0$ implies $u = 0$.
$(F_u) : F(u, u, 0, 0, u, u) = u^2(1 - 3k) \ge 0; \forall u > 0$.

Theorem 4. Let (X,d) be a metric space and S,T,I and $J : (X, d) \to (X, d)$ four mappings satisfying the inequality
(3.1) $F(d(Sx, Ty), d(Ix, Jy), d(Ix, Sx), d(Jy, Ty)d(Ix, Ty), d(Jy, Sx)) < 0$
for all x,y in X where F satisfies property (F_u).
Then S,T,I,J have at most one common fixed point.
Proof. Suppose that S,T,I and J have two common fixed points z and v with $z \ne v$. Then by (3.1) we have successively
$F(d(Sz, Tv), d(Iz, Jv), d(Iz, Sz), d(Jv, Tv), d(Iz, Tv), d(Jv, Sz) < 0$
$F(d(z, v), d(z, v), 0, 0, d(z, v), d(z, v)) < 0$ a contradiction of (F_u).

In this paper, using a combination of methods used in [1],[20] and [22] the results from Theorems 1-3 are improved by removing the assumption of continuity, relaxing compatibility to weakly compatibility property and replacing the completeness of the space with a set of four alternative conditions for functions satisfying an implicit relation.

4   Main results

Theorem 5. Let S,T,I and J be the self mappings of a metric space (X,d) such that
a) $S(X) \subset J(X)$ and $T(X) \subset I(X)$,
b) given $\varepsilon > 0$, there exists $\delta > 0$ such that
$\varepsilon \le max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), [d(Ix, Jy) + d(Jy, Tx)]/2\} < \varepsilon + \delta$ implies $d(Sx, Ty) < \varepsilon$ and
c) there exists $F \in \mathcal{F}_6$ such that inequality (3.1) holds for all x,y in X.
If one of S(X),T(X),I(X) and J(X) is a complete subspace of X, then

d) S and I have a coincidence point, e) T and J have a coincidence point.

Moreover, if the pairs (S,I) and (T,J) are weakly compatible, then S,T,I and J have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X, then since (a) holds we can define inductively a sequence

$\{Sx_0, Tx_1, Sx_2, Tx_3, ..., Sx_{2n}, Tx_{2n+1}, ...\}$

such that $y_{2n} = Sx_{2n} = Jx_{2n+1}$, $Y_{2n+1} = Tx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, 2, ...$

By [2 Lemma 2.2] it follows that $\{y_n\}$ is a Cauchy sequence in X.

Now suppose that J(X) is a complete subspace in X, then the sequence $y_{2n} = Jx_{2n+1}$, is a Cauchy sequence in J(X) and hence has a limit u.

Let $v \in J^{-1}u$, then $Jv = u$. Since y_{2n} is convergent and y_{2n+1} also converges to u. Setting $x = x_{2n}$ and $y = v$ in (3.1) we have

$F(d(Sx_{2n}, Tv), d(Ix_{2n}, Jv), d(Ix_{2n}, Sx_{2n}), d(Jv, Tv),$
$d(Ix_{2n}, Tv)d(Jv, Sx_{2n})) < 0.$

Letting n tend to infinity we obtain

$F(d(u, Tv), 0, 0, d(u, Tv), d(u, Tv), 0) \le 0$.

By (F_2) we have $u = Tv$. Hence J and T have a coincidence point. Since $T(X) \subset I(X)$, $u = Tv$ implies that $u \in I(X)$. Let $w = I^{-1}u$, then $Iw = u$.

Setting $x = w$ and $y = x_{2n+1}$ we obtain by (F_1) that $Sw = u$. Thus S and T have a coincidence point.

If one assumes that I(X) is complete then analoguous arguments establishes the earlier conclusion.

The remaining two cases are essentially the same as the previous cases. Indeed, if S(X) is complete, then by (a) $u \in S(X) \subset J(X)$.

Similarly, if T(X) is complete then $u \in T(X) \subset I(X)$. Then (d) and (e) are completely established.

By $u = Jv = Tv$ and weak compatibility of (J,T) we have

$$Tu = TJv = JTv = Ju .$$

By $u = Iw = Sw$ and weakly compatibility of (I,S) we have

$$Su = SIw = ISw = Iu .$$

By (3.1) we have successively

$F(d(Sw, Tu), d(Iw, Ju), d(Iw, Sw), d(Ju, Tu), d(Iw, Tu), d(Ju, Sw)) < 0$
$F(d(u, Tu), d(u, Tu), 0, 0, d(u, Tu), d(u, Tu)) < 0$

a contradiction of (F_u) if $u \ne Tu$. Thus $u = Tu$. Similarly we can show that $Su = u$.

Therefore, $u = Tu = Ju = Su = Iu$ and u is a common fixed point of S,T,I and J . The uniqueness of the common fixed point follows from Theorem 4.

Corolarry 1. Let S,T,I and J be the self mappings of a complete metric space satisfying conditions a),b),c) of Theorem 5. Then d) and e) hold. Moreover, if the pairs (S,I) and (T,J) are compatible (compatible of type (A), compatible of type (B), compatible of type (P)), then S,T,I and J have a unique common fixed point.

Proof. It follows by Theorem 5 and Remark4 1.

Corollary 2. Theorem 1.
Proof. It follows by Corollary 1 and Ex.1.

Remark 2. By Corollary 1 and Ex.3 we obtain Theorem 2 for $0 \leq k_1 < 1$ and $1 \leq k_2 < 2$.
By Corollary 1 and Ex.2 we obtain Theorem 3 for $0 \leq k < 1$.

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