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On Measure of Non-Compactness in Convex Metric Spaces

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Abstract. In this paper new results concerning the measure of non-compactness in convex metric spaces are presented.

1. **Introduction**

Let (X, d) be a metric space and A be a bounded subset of X.
Denote

$$\beta(A) = \inf\{\epsilon > 0 |\ A \text{ can be covered with a finite number of balls of radii smaller then } \epsilon\}.$$

This function is called the Hausdorff, measure of non-compactness. From definition one can directly obtain the following facts:

(1) $\beta(A) = 0 \Leftrightarrow A$ is totaly bounded;
(2) $\beta(A) = \beta(\bar{A})$;
(3) $A \subset B \Rightarrow \beta(A) \leq \beta(B)$;
(4) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$;
(5) $\beta(A \cap B) \leq \min\{\beta(A), \beta(B)\}$.

If our space is Banach this function has some additional properties connected with the linear structure:

6) $\beta(A + B) \leq \beta(A) + \beta(B)$;
7) $\beta(\alpha A) = |\alpha|\beta(A), \quad \alpha \in \mathbb{R}$;
8) $\beta(convA) = \beta(A)$, ($conv\ A$ is a convex hull of A).

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It is well known that in the fixed point theory the relation 8) i.e.

$$\beta(convA) = \beta(A)$$

is of great importance. In locally convex metric spaces relation 8) is thrue. But, if we considered topological vector space is not locally convex space we can not prove, in general, that $\beta(convA) = \beta(A)$ (see [2] for example). Further, one can introduce a concept of convexity in metric space in abstract form and study the properties of such spaces – called *convex metric spaces.* At first it was done by W. Takahashi [3].

Definition 1. *Let X be a metric space and I be the closed unit interval. A mapping $W : X \times X \times I \to X$ is said to be convex structure on X if for all $x, y, u \in X$, $\lambda \in I$,*

$$d(u, W(x, y, \lambda) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

X together with a convex structure is called a Takahaski convex metric space (X, d, W) or TCS.

Any convex subset of a normed space is a convex metric space with $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$. There are non-trvial examples of TCS.

Example 1. *Let I be the unit interval $[0, 1]$ and X be the family of closed intervals $[a_i, b_i]$ such that $0 \leq a_i \leq b_i \leq 1$. For $I_i = [a_i, b_i], I_j = [a_j, b_j]$ and $\lambda (0 \leq \lambda \leq 1)$, we define a mapping W by $W(I_i, I_j, \lambda) = [\lambda a_i + (1 - \lambda)a_j, \lambda b_i + (1 - \lambda)b_j]$ and define a metric d in X by the* Hausdorff distance*, i.e.*

$$d(I_i, I_j) = \sup_{a \in I}\{|\inf_{b \in I_i}\{|a - b|\} - \inf_{c \in I_j}\{|a - c|\}|\}.$$

Definition 2. *Let X be a convex metric space. A nonempty subset K of X is said to be convex if and only if $W(x, y, \lambda) \in K$ whenever $x, y \in K$ and $\lambda \in I$.*

Takahashi has shown ([3]) that open and closed balls are convex, and that the arbirary intersection of convex sest is convex too.

For arbitrary $A \subset X$ let

$$\widetilde{W}(A) := \{W(x, y, \lambda) \ : \ x, y \in A, \ \lambda \in I\}. \tag{1}$$

It is easy to see that $\widetilde{W} : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ is a mapping with properties:

i) $A \subset \widetilde{W}(A)$, for $A \subset X$;
ii) $A \subset B$ implies $\widetilde{W}(A) \subset \widetilde{W}(B)$, for any $A, B \in \mathcal{P}(\mathcal{X})$;
iii) $\widetilde{W}(A \cap B) \subset \widetilde{W}(A) \cap \widetilde{W}(B)$, for any $A, B \in \mathcal{P}(\mathcal{X})$.

Using this notation we say that K is convex iff $\widetilde{W}(K) \subset K$.

Now, let us recall:

Definition 3. *The convex hull of a set A ($A \subset X$) is the intersection of all convex sets in X containing A and it is denoted by conv A.*

It is obvious that if A is a convex set then $\widetilde{W}^n\,(A) = \widetilde{W}(\widetilde{W}(\ldots \widetilde{W}(A)\ldots) \subset A$ for any $n \in N$. If we set:

$$A_n = \widetilde{W}^n(A), \quad (A \subset X),$$

the sequence $\{A_n\}_{n\in N}$ is increasing so $\limsup A_n$ exists and

$$\limsup A_n = \liminf A_n = \lim A_n = \cup_{n=1}^{\infty} A_n.$$

Proposition 1. *Let X be a convex mertic space and $(A \subset X)$. Then*

$$convA = \lim A_n = \cup_{n=1}^{\infty} A_n.$$

Proposition 2. [1] *For any subset A of (X, d, W)*

$$\delta(convA) = \delta(A),$$

$\delta(A) = sup_{x,y\in A} d(x, y)$.

Definition 4. *TSC X has property P if for every x_1, x_2, y_1, y_2, $\in X$, $\lambda \in I$*

$$d(W(x_1, x_2, \lambda), W(y_1, y_2, \lambda)) \le \lambda d(x_1, y_1) + (1 - \lambda)d(x_2, y_2).$$

Obviously in normed space it is always satisfied.

Definition 5. *TSC X has property Q if for any finite subset $A \subset X$ $convA$ is compact set.*

L. Talman [4] introduced a new notion of convex structure for metric spaces based on Takahashi notion but with some aditional properties- so called *strong convex structure (SCS)*. In SCS condition Q is always satisfied so it seems to be "natural."

2. Main result

Theorem 1. *Set (X, d, W) be TSC with propetier P and Q. Then for any bounded subset $A \subset X$.*

$$\beta(convA) = \beta(A).$$

Proof. Let A be a bounded subset of X. Let us prove that

$$\beta(convA) \le \beta(A).$$

From definition of measure of non-compactness it follows that for any $\varepsilon > 0$ there exists finite subset $B = \{x_1, x_2, \ldots, x_n\}$ such that

$$A \subset \bigcup_{i=1}^{n} L(x_i, \beta(A) + \frac{\varepsilon}{2}). \tag{2}$$

Since convex hull for any finite subset is compact (property Q), there is $\{u_1, u_2, \ldots u_s\} \subset B$ such that

$$convB \subset \bigcup_{i=1}^{s} L(u_i, \frac{\varepsilon}{2}). \tag{3}$$

For any $y \in convA$ find $n_0 \in \mathbb{N}$ such that

$$y \in A_{n_0} = \widetilde{W}(A_{n_0-1}) = \quad \widetilde{W}^{n_0}(A).$$

But it means that

$$y = W(y_1^1, y_2^1, \lambda_1^1)$$

for some $y_1^1, y_2^1 \in A_{n_0-1}$, $\lambda_1^1 \in I$ and so on for y_1^1, y_2^1. After not more than n_0 steps we have that y is image of $\widetilde{W}^{n_0}$ for some $y_1^{n_0}, y_2^{n_0}, \ldots, y_{2^{n_0}}^{n_0}$ from A and $\lambda_1^{n_0}, \lambda_2^{n_0}, \ldots, \lambda_{2^{n_0}-1}^{n_0}$ from I. Further, for any $y_k^{n_0}, k = 1, 2, \ldots, 2^{n_0}$ there is $x_{i(k)} \in \{x_1, x_2, \ldots, x_n\}$ with property that

$$d(y_k^{n_0}, x_{i(k)}) < \beta(A) + \frac{\varepsilon}{2}.$$

Now we make image of $x_{i(1)}, x_{i(2)}, \ldots, x_{i(2^{n_0})}$ and $\lambda_1^{n_0}, \lambda_2^{n_0}, \ldots, \lambda_2^{n_0-1}$ on the same way (but in oposite direction) as for y. Set it be x. Obviously $x \in conv\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{s} L(u_i, \frac{\varepsilon}{2})$ (see (3)) so there exists $k_0 \in \{1, 2, \ldots, n\}$ such that.

$$x \in L(U_{k_0}, \frac{\varepsilon}{2}).$$

Using properti Q and (2) one can chek that

$$d(x, y) \leq \sum_{k=1}^{2n_0} \lambda_i d(y_k^{n_0}, x_{i(k)}) < \beta(A) + \frac{\varepsilon}{2}.$$

Finaly we have that

$$d(y, u_{k_0}) \leq d(y, x) + d(x, u_{k_0}) < \beta(A) + \varepsilon.$$

Since the inequality

$$\beta(A) \leq \beta(convA)$$

is valid we prove that

$$\beta(convA) = \beta(A).$$

□

3. Application

Recall that a multifunction $F : X \to Y$ is a function which assigns to each $x \in X$ a nonempty subset $F(x)$ of Y, and denote $F : X \to 2^Y$.

If $F : X \to 2^Y$ and $A \subset X$ then by $F(A)$ we mean the set $\{y \in Y | y \in F(x), x \in A\}$. The graph of F is the set $\{(x, y) | y \in F(x), x \in X\}$. A fixed point of $F : X \to 2^X$ is a point satisfies $x \in F(x)$.

Definition 6. *Let (X, d) be metric space, m measure of non-compactness on X and $F : X \to 2^X$. We say that F is condensing if for every bounded subset $A \subset X$, the relation $m(A) > 0$ implies that $m(F(A)) < m(A)$. In our investigation let m be the Hausdorff measure of non-compactnes β.*

Theorem 2. *Let (X, d, W) be a complete TCS with continuous structure W and properties $(P), (Q)$, and let $F : X \to 2^X$ be a condensing function with convex values, closed graph and bounded range. Then F has a fixed point.*

Proof. Simillary as in [3] using Theorem 1. □

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