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SKEW EXACTNESS AND RANGE-KERNEL ORTHOGONALITY

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Abstract. Range-kernel orthogonality is set in a context of skew exactness, in particular for elementary operators on bimodules.

Suppose $(S, T) : X \to Y \to Z$, bounded linear operators between Banach spaces (by which we mean $T : X \to Y$ and $S : Y \to Z$, so that ST is defined): then the null space $S^{-1}(0)$ and the range TX are subsets of the same space Y and can be compared. Thus $ST = 0$ iff $TX \subseteq S^{-1}(0)$, and the pair (S, T) is "exact" if the opposite inclusion holds. When they are either disjoint (intersection $\{0\}$), or add up to the whole space Y, we shall think of the pair (S, T) as in some sense "skew exact" ([12] §10.9; [13];[14]). Among variations on this theme lies a certain "range-kernel orthogonality", based on James' Banach space orthogonality. In this note we look at this, in particular for "elementary operators". We begin by noticing how certain kinds of operator comparison ([12] §§10.1,10.2) transmit corresponding kinds of skew exactness:

1. Theorem *If* $(S, T) : X \to Y \to Z$ *and* $(S', T') : X' \to Y' \to Z'$ *satisfy*

1.1 $$WS' = SV\ ,\ VT' = TU\ ,\ V'V = I$$

then there is implication

1.2 $$T = RST \Longrightarrow T' = R'S'T'\ .$$

If instead

1.3 $$S'^{-1}(0) \subseteq (SV)^{-1}(0)\ ,\ VT'X' \subseteq TX\ ,\ V^{-1}(0) = \{0\}$$

then there is implication

1.4 $$S^{-1}(0)_\cap T(X) = \{0\} \Longrightarrow S'^{-1}(0)_\cap T'(X') = \{0\}\ .$$

If instead

1.5 $$\|SV(\cdot)\| \le h\|S'(\cdot)\|\ ,\ VT'X' \subseteq TX\ ,\ \|\cdot\| \le \ell\|V(\cdot)\|$$

then there is implication

1.6 $$\|T(\cdot)\| \le k\|ST(\cdot)\| \Longrightarrow \|T'(\cdot)\| \le k'\|S'T'(\cdot)\|\ .$$

Proof. For (1.2) argue

$$VT' = TU = RSTU = RSVT' = RWS'T'\ ,\ \Longrightarrow T' = (V'RW)S'T'\ .$$

For (1.4) argue

$$S'y' = 0 \Longrightarrow SVy' = 0 \Longrightarrow Vy' \notin TX \Longrightarrow y' \notin V^{-1}TX \supseteq T'X'\ .$$

Finally, for (1.6),

$$\|T'x'\| \le \ell\|VT'x'\| = \ell\|Tx\| \le k\ell\|STx\| = k\ell\|SVT'x'\| \le k\ell h\|S'T'x'\| \bullet$$

Dually,

2. Theorem *If* $(S,T) : X \to Y \to Z$ *and* $(S',T') : X' \to Y' \to Z'$ *satisfy*

2.1 $$WS = S'V\ ,\ VT = T'U\ ,\ VV' = I$$

then there is implication

2.2 $$S = STR \Longrightarrow S' = S'T'R'\ .$$

If instead

2.3 $$V(S^{-1}(0)) \subseteq {S'}^{-1}(0)\ ,\ VT(X) \subseteq T'(X')\ ,\ VY = Y'$$

then there is implication

2.4 $$S^{-1}(0) + T(X) = Y \Longrightarrow {S'}^{-1}(0) + T'(X') = Y'\ .$$

If instead

2.5 $$V(S^{-1}(0)) \subseteq {S'}^{-1}(0)\ ,\ VTx = T'x' \text{ with } \|x'\| \le h\|x\|\ ,\ V \text{ open}$$

then there is implication

2.6 $$Sy = STx \text{ with } \|x\| \le k\|y\| \Longrightarrow S'y' = S'T'x' \text{ with } \|x'\| \le k'\|y'\|\ .$$

Proof. For (2.2) argue exactly as for (1.2), reversing products. For (2.4) argue, with $Sw = 0$,

$$y' \in Y' \Longrightarrow y' = Vy = V(w + Tx) = Vw + T'Ux' \in {S'}^{-1}(0) + T'(X')\ .$$

Finally, for (2.6),

$$\begin{aligned} &y' \in Y' \Longrightarrow \\ &S'y' = S'Vy = S'V(w + Tx) = S'T'x' \text{ with } \|y\| \le \ell\|y'\|\ , \\ &Sw = 0 \text{ and } \|x\| \le k\|y\| \bullet \end{aligned}$$

Our "weak orthogonality" comes from James' Banach space orthogonality for subspaces:

3. Definition *If* $(S, T) : X \to Y \to Z$ *and* $k > 0$ *we declare*

3.1 $$S\angle_k T \Longleftrightarrow S^{-1}(0)\angle_k T(X) \Longleftrightarrow \big(y \in S^{-1}(0) \Longrightarrow \|y\| \le k\text{dist}(y, T(X))\big)\ ,$$

and call S *weakly orthogonal to* T*, written* $S\angle T$*, equivalently* $S^{-1}(0)\angle T(X)$*, provided*

3.2 $$\exists k > 0,\ S\angle_k T\ .$$

If (3.1) holds with $k = 1$ we shall say that S is orthogonal to T, written $S \perp T$. If (3.1) with $k = 1$, or (3.2), holds with $S = T$ we shall call T orthogonal, or weakly orthogonal .

Weak orthogonality lies ([14] (2.7)) between the conditions (1.4) and (1.6), and is transmitted by a hybrid of the conditions (1.3) and (1.5):

4. Theorem *If $(S,T) : X \to Y \to Z$ and $(S',T') : X' \to Y' \to Z'$ satisfy*

4.1 $$S'^{-1}(0) \subseteq (SV)^{-1}(0)\ ,\ VT'X' \subseteq TX\ ,\ \|\cdot\| \le \ell\|V(\cdot)\|$$

then there is implication

4.2 $$S\angle T \Longrightarrow S'\angle T'\ .$$

Proof. If $y \in S^{-1}(0) \Longrightarrow \|y\| \le k\mathrm{dist}(y,TX)$ then $S'y' = 0 \Longrightarrow SVy' = 0$ and hence if $S'y' = 0$ then

$$\begin{aligned}\|y'\| \le \ell\|Vy'\| \le \ell k\mathrm{dist}(Vy',TX) \le \ell k\mathrm{dist}(Vy',VT'X')\\ \le \ell k\|V\|\mathrm{dist}(y',T'X')\ \bullet\end{aligned}$$

Under the conditions (1.1) we can reverse the implication (4.2) if there are "approximate inverse intertwinings" in the sense of Shulman/Turowska [22]:

5. Theorem *If $(S,T) : X \to Y \to Z$ and $(S',T') : X' \to Y' \to Z'$ satisfy*

5.1 $$\begin{aligned}WS' = SV\ ,\ VT' = TU\ ,\ S^{-1}(0) \subseteq V(S'^{-1}(0)),\\ V'_\alpha V \to I\ ,\ V'_\alpha T - T'U'_\alpha \to 0\ ,\end{aligned}$$

with convergence in the strong operator topology, then there is inclusion

5.2 $$V^{-1}(TX) \subseteq \mathrm{cl}\ T'(X')$$

and implication

5.3 $$S'^{-1}(0)_\cap\mathrm{cl}\ T'X' = \{0\} \Longrightarrow S^{-1}(0)_\cap TX = \{0\}\ .$$

If in addition

5.4 $$\sup_\alpha \|V'_\alpha\| < \infty \text{ and } \mathrm{cl}\ UX = X'$$

then

5.5 $$S'\angle T' \Longrightarrow S\angle T\ .$$

Proof. This is the argument of Shulman/Turowska ([22] Theorem 6.1, Corollary 6.2):

$$Vy = Tx \Longrightarrow y' = \lim_\alpha V'_\alpha Tx = \lim_\alpha T'U'_\alpha x \in \text{cl } T'X'\ ,$$

giving (5.2), while for (5.3)

$$V(S'^{-1}(0))_\cap TX = V\big(S'^{-1}(0)_\cap V^{-1}(TX)\big) \subseteq V\big(S'^{-1}(0)_\cap \text{cl } TX\big) = V(\{0\})\ .$$

Finally for (5.5) $y' \in S'^{-1}(0)$ gives for arbitrary $x' \in X'$

$$\begin{aligned}\|Vy'\| &\le k\ \|V\|\ \|y' + T'x'\| = k\|V\| \lim_\alpha \|V'_\alpha(Vy' + VT'x')\| \\ &\le k\|V\| \sup_\alpha \|V'_\alpha\|\ \|Vy' + TUx'\|\end{aligned}$$

and hence

$$\|Vy'\| \le h\ \text{dist}(Vy', TUX') = h\ \text{dist}(Vy', TX)\ \bullet$$

Specialising to $X = Y = Z$ and $X' = Y' = Z'$, if S, T, S' and T' satisfy

5.6 $$\begin{aligned}VS' = SV\ ,\ VT' = TV\ ,\ V'_\alpha S - S'V'_\alpha \to 0\ ,\ V'_\alpha T - T'V'_\alpha \to 0\ ,\\ V'_\alpha V \to I\ ,\ VV'_\alpha \to I\end{aligned}$$

then

5.7 $$\begin{aligned}\|T'(\cdot)\| \le k\|S'(\cdot)\| \Longrightarrow \\ S^{-1}V(Y') \subseteq T^{-1}V(Y') \textit{ and } \|V^{-1}T(\cdot)\| \le k\|V^{-1}S(\cdot)\|\ .\end{aligned}$$

This is Shulman/Turowska ([22] Lemma 6.4): for the first implication

$$Sy = Vy' \Longrightarrow y' = \lim_\alpha V'_\alpha Sy = \lim_\alpha S'V'_\alpha y$$

giving by completeness and cauchyness

$$\exists x' = \lim_\alpha T'V'_\alpha y = \lim_\alpha V'_\alpha Ty \textit{ with } \|x'\| \le k\|y'\| \textit{ and } Vx' = Ty\ ,$$

while

$$\|V^{-1}Ty\| = \|x'\| \le k\|y'\| = k\|V^{-1}Sy\| \ .$$

The archetypical example of orthogonality $S \perp T$ occurs when $Z = X$ and Y are Hilbert spaces and $S = T^*$ is the adjoint of T:

6. Definition *If $* : B \to BL(Y, X)$ is an involution defined on a linear subspace $B \subseteq BL(X, Y)$ we shall describe $T \in B$ as *-orthogonal provided*

6.1 $$T^* \perp T \ ,$$

*weakly *-orthogonal provided*

6.2 $$T^* \angle T \ ,$$

*and ultra weakly *-orthogonal provided*

6.3 $$(T^*T)^{-1}(0) \subseteq T^{-1}(0) \ .$$

If in particular $Y = X$ and $B^* = B$ we shall call $T \in B$ hyponormal provided

6.4 $$\|T^*(\cdot)\| \le \|T(\cdot)\| \ ,$$

weakly hyponormal provided there is $k > 0$ for which

6.5 $$\|T^*(\cdot)\| \le k \ \|T(\cdot)\| \ ,$$

and Fuglede provided there is inclusion

6.6 $$T^{-1}(0) \subseteq T^{*-1}(0) \ .$$

Finally we call $T \in B$ normal provided

6.7 $$TT^* = T^*T \ .$$

Obviously each of the first three conditions implies the next, and also each of the second three. If for example X and Y are Hilbert spaces and $*$

is the usual adjoint, defined on the whole space $B = BL(X, Y)$, then every operator $T \in B$ satisfies (6.1): we recall

6.8 $$\|Tx\|^2 \leq \|T^*Tx\| \; \|x\| \; (x \in X)$$

and

6.9 $$x \in X \; , \; T^*y = 0 \Longrightarrow \|Tx + y\|^2 = \|Tx\|^2 + \|y\|^2 \; .$$

On Hilbert space also weakly hyponormal operators satisfy a strengthened form of the condition (6.5):

6.10 $$T^* = UT \; .$$

We cannot however strengthen (6.1) to the analogue of the left hand side of (1.6): for example ([12] (10.5.2.9)) take $X = Y = \ell_2$ and set $(Tx)_n = n^{-1}x_n$.

7. Theorem *If* $* : B \to B \subseteq B(X)$ *and if* $T^* \in B$ *is ultra weakly *-orthogonal then*

7.1 $$T \textit{ normal } \Longrightarrow T \textit{ Fuglede} \; .$$

If $T : X \to Y$ *between Hilbert spaces and* $S : Y \to Z$ *into a Banach space then*

7.2 $$S \perp T \Longleftrightarrow S^{-1}(0) \subseteq T^{*-1}(0) \; .$$

Also

7.3 $$T = UT^*T \Longleftrightarrow T(X) = \mathrm{cl}\; T(X) \; .$$

Proof. For (7.1) we argue

$$T^{-1}(0) \subseteq (T^*T)^{-1}(0) = (TT^*)^{-1}(0) \subseteq T^{*-1}(0) \; .$$

For (7.2) recall

$$T(X)^\perp = T^{*-1}(0) \; .$$

If $T : X \to Y$ has closed range then the left hand side of (7.3) holds with U^* the "Moore-Penrose inverse" of T. Conversely if $Q^* = Q = Q^2$ is the orthogonal projection on Y for which $Q(Y) = \mathrm{cl}(TX)$ and $T = UT^*T$ then $Q = UT^*Q = UT^*$ so that

$$TX \supseteq TU^*Y = QY = \mathrm{cl}\, TX \bullet$$

From (7.2) it follows, if $T \in B(X)$ for Hilbert space X, that

7.4 $$T \textit{ Fuglede } \Longleftrightarrow T \textit{ orthogonal} .$$

Shulman/Turowska [22] describes (6.3) as the "non commutative Fuglede theorem" for Hilbert space operators. When $Y = X$ is a Banach space there is an involution derived from the concept of *numerical range* [4],[6]:

8. Definition *If A is a Banach algebra define the numerical range of $a \in A^n$ by means of states $\varphi \in A^\dagger$:*

8.1 $$V_A(a) = \{\varphi(a) : \|\varphi\| = 1 = \varphi(1)\} \subseteq \mathbf{C}^n .$$

The Hermitian elements of A are those with real numerical range:

8.2 $$\mathrm{Re}(A) = \{a \in A : V_A(a) \subseteq \mathbf{R}\} = \{a \in A : t \in \mathbf{R} \Longrightarrow \|e^{ita}\| = 1\} .$$

Now write

8.3 $$\mathrm{Reim}(A) = \mathrm{Re}(A) + i\ \mathrm{Re}(A) ,$$

and define

8.4 $$(h + ik)^* = h - ik \textit{ whenever } (h, k) \in \mathrm{Re}(A)^2 .$$

The equivalence of the two conditions in (8.2) is ([4] Lemma 5.2) not trivial. Also ([4] Lemma 5.7) if $a = h + ik \in A$ with hermitian h and k then h and k are determined uniquely, so that (8.4) is a good definition. Since $V_A(\alpha a + \beta b) \subseteq \alpha V_A(a) + \beta V_A(b)$ it is clear that real linear combinations of hermitian elements are hermitian.

With the involution (8.4) normality (6.7) of $a = h+ik$ occurs when h and k commute. A theorem of Palmer ([18] Lemma 2.7; [5] Proposition 2) says, for normal $T = H + iK \in B(X)$, that if all products H^pK^q are hermitian then (6.4) holds with equality. An example of Anderson and Foias ([2]

Example 5.9) warns us that this can easily fail, even in 2 dimensions.

Sinclair's Theorem ([23] Proposition 1; [11] Corollary 7) says that boundary points of the numerical range breed orthogonality: if $T : X \to X$ for a Banach space X then

8.5 $$0 \notin \mathrm{int}\, V_{B(X)}(T) \Longrightarrow T \textit{ orthogonal} .$$

Fong's result ([11] Lemma 3, Theorem A) is that on Banach spaces normal operators are orthogonal and Fuglede:

8.6 $$T \textit{ normal} \Longrightarrow T^{*-1}(0) = T^{-1}(0) \perp T(X) .$$

Thus on two counts hermitian elements are orthogonal.

When $a \in A^n$ and $b \in B^n$ are tuples of Banach algebra elements and

8.7 $$T = L_a \circ R_b = R_b \circ L_a : x \mapsto \sum_j a_j x b_j$$

is an "elementary operator", defined on a Banach (A, B)-bimodule M, we look for such orthogonality. Notice that if $a \in A^n$ and $b \in B^n$ then

8.8 $$V_{B(M)}(L_a) \subseteq V_A(a) \ , \ V_{B(M)}(R_b) \subseteq V_B(b) \ ,$$

with equality if $M = A$ or $M = B$:

8.9 $$\Phi \in B(M)^\dagger \to \varphi : a \mapsto \Phi(L_a) \ ; \ \varphi \in A^\dagger \to \Phi : T \mapsto \varphi(T1) \ .$$

Hence if $T = L_c - R_d : x \mapsto cx - xd$ with hermitian, or normal, $c \in A$ and $d \in B$ then T is again hermitian, or normal. It follows that (8.6) applies:

8.10
$$c, d \textit{ normal} \Longrightarrow L_c - R_d \textit{ normal} \Longrightarrow L_c - R_d \textit{ orthogonal and Fuglede} ;$$

this incorporates an extension of what is known as the Putnam-Fuglede theorem. One consequence is that if $a = h + ik \in A$ is normal then the commutant of a is the same as the commutant of the pair (h, k); it then follows that the sum of two commuting normal elements is again normal.

Unfortunately (8.10) is not [2] clear for products: it does not generally follow that hermitian or normal tuples a, b lead to hermitian or normal $L_a \circ R_b$. Shulman/Turowska ([22] Proposition 9.8) has an example of commuting normal $a \in A^n$, $b \in A^n$ for which $L_a \circ R_b$ is not a Fuglede operator.

One way to achieve normality of $L_a \circ R_b$ is for M to be a Hilbert space:

9. Theorem *If the (A, B)-bimodule M is a Hilbert space and $a \in A^n$ and $b \in B^n$ are commuting tuples of normal elements then*

9.1 $$L_{a^*} \circ R_{b^*} \perp L_a \circ R_b \ .$$

Proof. If $a \in A^n$ and $b \in B^n$ are tuples of complex combinations of hermitian elements then, acting on the Hilbert space M with the standard involution $*$,

9.2 $$(L_a \circ R_b)^* = L_{a^*} \circ R_{b^*} \ .$$

In particular if the a_j and b_i are normal then so are the L_{a_j} and R_{b_i}, and hence also $T = L_a \circ R_b$: now (6.1) applies •

Theorem 9 applies in particular [28] if $A = B = B(X)$ for a Hilbert space X and $M = C_2(X)$ is the Schatten class of Hilbert-Schmidt operators. Theorem 4 suggests that if $L_a \circ R_b$ is orthogonal on $A = B = B(X)$ then it is also orthogonal on the ideal $M = C_2(X)$: more pertinent is Theorem 5 which suggests that if $L_a \circ R_b$ is orthogonal on $C_2(X)$ then it is orthogonal on $B(X)$.

The context of Theorem 9 can be marginally extended: Turnsek ([27] Theorem 2.8) has the result that if M is one of the Schatten ideals $C_p(X)$ in the algebra $A = B(X)$ for a Banach space X then, for commuting normal pairs $a \in A^2$ and $b \in A^2$ and a Hilbert space X, there is inclusion

9.3 $$\mathrm{cl}_M(M_\cap(L_a \circ R_b)(A)) \subseteq \mathrm{cl}_M(L_a \circ R_b)(M) \ .$$

Without normality (9.3) may fail ([27] Example 2.9); with $A = B(X)$ for Hilbert space X and $M = C_1(X)$ the trace class take

9.4 $$T = I - L_u R_v \textit{ with } u, v \textit{ the forward and backward shifts } :$$

claim

9.5 $$1 - uv = T(1) \in T(A)_\cap M \textit{ and } 1 - uv \notin \mathrm{cl}_1\ T(M) \ .$$

Indeed there is duality between A and M implemented by the mapping

9.6 $$a \mapsto a^\wedge \in M^\dagger : \Sigma_j \varphi_j \odot \xi_j \mapsto \Sigma_j \varphi_j(a\xi_j) \ ,$$

under which the Banach space dual of the mapping T is given by

9.7 $$(I - L_u R_v)^\dagger = I - L_v R_u : A \to A \ .$$

Evidently

9.8 $$\{\lambda : \lambda \in \mathbf{C}\} \subseteq (I - L_v R_u)^{-1}(0) \subseteq A$$

while

9.9 $$\lambda \neq 0 \Longrightarrow \lambda \notin \mathrm{cl}_M (I - L_u R_v)(M) \ .$$

When the involution is derived from numerical range there is [17] an alternative concept of "hyponormal":

10. Definition *Call $a \in A$ positive if it has positive real numerical range*

10.1 $$V_A(a) \subseteq [0, \infty) \ ,$$

and hyponormal if it has a positive self commutator:

10.2 $$a \in \mathrm{Re}(A) + i\ \mathrm{Re}(A) \ \textit{with}\ V_A(a^* a - a a^*) \subseteq [0, \infty) \ .$$

In this sense Mattila ([17] Theorem 4.3) and Shaw ([20] §3) have partially extended (8.10):

10.3 $$c, d^* \ \textit{hyponormal} \implies L_c - R_d \ \textit{hyponormal} \ .$$

Part of the argument is a partial extension of (8.8): if $A = B(X)$ and $B = B(Y)$ then ([20] Theorem 1.1)

10.4 $$X^\dagger \odot Y \subseteq M \Longrightarrow V_{B(M)}(L_c - R_d) = V_A(c) - V_B(d)\ .$$

The non trivial part of the argument passes through the "spatial" numerical range of $L_c - R_d$: if $\|f\| = \|x\| = f(x) = 1$ and $\|g\| = \|y\| = g(y) = 1$ define $\Phi \in B(M)^\dagger$ by setting

$$\Phi(T) = f(T(g \odot x))y = (f \odot y)(T(g \odot x)))\ ,$$

so that $\Phi(L_c - R_d) = f(cx) - g(yd)$. For (10.3) we now observe

10.5 $$(L_c - R_d)^*(L_c - R_d) - (L_c - R_d)(L_c - R_d)^* = L_a - R_b \\ \text{with } a = c^*c - cc^*\ ,\ b = d^*d - dd^*\ .$$

Shulman has the result that if $a \in A^n$ and $b \in A^n$ are commuting tuples of normal C^*-algebra elements and $T = L_a \circ R_b$ then there is implication

10.6 $$T^{-1}(0)_\cap T(X) = \{0\} \Longrightarrow T \text{ Fuglede }\ :$$

the involution here is given by taking (9.2) as the definition of its left hand side, hoping that this is good. Keckic [15] and Turnsek [25] have, for commuting pairs of normal operators $a, b \in A^2$,

10.7 $$L_a^{-1}(0)_\cap R_b^{-1}(0) = \{0\} \Longleftrightarrow L_a \circ R_b \text{ orthogonal }.$$

Part of the argument is the observation that if c and d are commuting normals with invertible d then, with $a = (c, d)$ and $b = (d, -c)$,

$$L_a \circ R_b = L_c R_d - L_d R_c = L_d(L_{d^{-1}c} - R_{d^{-1}c})R_d\ .$$

More generally if $a \in A^n$ and $c \in A^n$ are "similar", and also $b \in B^n$ and $d \in B^n$, then so are $L_a \circ R_b$ and $L_c \circ R_d$:

10.8 $$uc_j = a_j u \text{ and } vd_j = b_j v\ (j = 1, 2, \dots, n) \Longrightarrow \\ L_u(L_c \circ R_d)R_v = R_v(L_a \circ R_b)L_u\ .$$

Thus if $u \in A^{-1}$ and $v \in B^{-1}$ are invertible so that also L_u and R_v are invertible in $B(M)$, then (1.1) is satisfied when $S = T = L_a \circ R_b$ and $S' = T' = L_c \circ R_d$. Another reduction is that

$$c = \begin{bmatrix} a_1 & 0 \\ 0 & b_2 \end{bmatrix}\ ,\ d = \begin{bmatrix} -a_2 & 0 \\ 0 & b_1 \end{bmatrix}\ ,\ w = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$$

gives

10.9
$$(L_cR_d - R_cL_d)(w) = \begin{bmatrix} 0 & (L_a \circ R_b)(x) \\ 0 & 0 \end{bmatrix} \ \textit{with}$$
$$L_c^{-1}(0) \cap R_d^{-1}(0) = \begin{bmatrix} L_a^{-1}(0) \\ R_b^{-1}(0) \end{bmatrix} .$$

It is an interesting problem how or whether numerical range hyponormality (11.2) relates to conditions (6.4) and (6.6). It would be tempting to try and extend the Fong argument for the second part of (8.6) from normal to hyponormal T, which would then offer an extension of (10.7) to commuting hyponormal pairs a, b^* in A^2.

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