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A LITTLEWOOD-PALEY TYPE INEQUALITY FOR HARMONIC FUNCTIONS IN THE UNIT BALL OF $\mathbb{R}^N$

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Abstract. It is proved the following: If u is a function harmonic in the unit ball $B \subset \mathbb{R}^N$, and $0 < p < 1$, then there holds the inequality

$$\sup_{0<r<1} \int_{\partial B} |u(ry)|^p \, d\sigma \le |u(0)|^p + C_{p,N} \int_B (1-|x|)^{p-1} |\nabla u(x)|^p \, dV(x)\,.$$

In the case $p > (N-2)/(N-1)$, this was proved by Stević [17].

Let $\mathbb{R}^N$ $(N \ge 2)$ denote the N-dimensional Euclidean space. In [17], Stević proved that if u is a function harmonic in the unit ball $B \subset \mathbb{R}^N$, and $\frac{N-2}{N-1} \le p < 1$, then there holds the inequality

$$\sup_{0<r<1} M_p^p(r,u) \le C_1 |u(0)|^p + C_2 \int_B (1-|x|)^{p-1} |\nabla u(x)|^p \, dV(x)\,. \tag{1}$$

Here dV denotes the Lebesgue measure in $\mathbb{R}^N$ normalized so that $V(B) = 1$, and as usual

$$M_p^p(r,u) = \int_{\partial B} |u(ry)|^p \, d\sigma\,,$$

where $d\sigma$ is the normalized surface measure on the sphere ∂B. It is the aim of this note to remove the strange condition $(N-2)/(N-1) \le p < 1$. This condition appears in [17] because the proof in the paper is based on the fact, due Stein and Weiss [16, 15], that $|\nabla u|^p$ is subharmonic for $p \ge (N-2)/(N-1)$. Our result is slightly stronger than (1):

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Theorem 1. *If u is a function harmonic in B, and $0 < p < 1$, then there holds the inequality*

$$\sup_{0<r<1} M_p^p(r,u) \le |u(0)|^p + C\int_B (1-|x|)^{p-1}|\nabla u(x)|^p\, dV(x)\,, \tag{2}$$

where C is a constant depending only on p and N.

In the case $N = 2$, this theorem was proved by Flett [2]. Inequality (2) holds for $1 < p < 2$ as well, while if $p > 2$, then there holds the reverse inequality; these inequalities are due to Littlewood and Paley [6]. Elementary proofs of the Littewood-Paley inequalities are given in [12] and [7, 14] ($p > 2$).

Observe that if $u > 0$ in B, and $0 < p < 1$, then (2) is completely trivial because then function u^p is superharmonic and therefore

$$\sup_{0<r<1} M_p^p(r,u) \le |u(0)|^p\,.$$

Thus (2) shows in particular how much $|u|^p$ is far from being superharmonic.

Our proof of Theorem 1 is based on a fundamental result of Hardy and Littlewood [3] and Fefferman and Stein [1] on subharmonic behavior of $|u|^p$. We state this result in the following way.

Lemma 1. *If $U \ge 0$ is a function subharmonic in $B(a, 2\varepsilon)$ $(a \in \mathbb{R}^N, \varepsilon > 0)$, then there holds the inequality*

$$\sup_{x\in B(a,\varepsilon)} U(x)^p \le C\varepsilon^{-N}\int_{B(a,2\varepsilon)} U^p dV, \qquad 0 < \varepsilon < 1\,, \tag{3}$$

where C depends only on p, N.

Here $B(a,r)$ denotes the ball of radius r centered at a. For simple proofs of Lemma 1 we refer to [9, 13], and for generalizations to various classes of functions, we refer to [4, 5, 8, 10, 11]. From Lemma 1 we shall deduce the following crucial fact:

Lemma 2. *Let $r_j = 1 - 2^{-j}$ for $j \ge 0$, and $r_{-1} = 0$. If $0 < p < 1$ and u is harmonic in B, then there holds inequality*

$$M_p^p(r_{j+1},u) - M_p^p(r_j,u) \le C\int_{r_{j-1}\le|x|\le r_{j+2}} (1-|x|)^{p-1}|\nabla u(x)|^p\, dV(x)\,, \quad j \ge 0\,,$$

where C depends only on p and N.

Proof. We start from the inequality

$$M_p^p(r_{j+1},u) - M_p^p(r_j,u) \le \int_S |u(r_{j+1}y) - u(r_j y)|^p\, d\sigma(y)\,. \tag{4}$$

By Lagrange's theorem,

(5)
$$|u(r_{j+1}y)-u(r_jy)| \le (r_{j+1}-r_j) \sup_{r_j<r<r_{j+1}} |\nabla u(ry)| \le 2^{-j} \sup_{r_j<r<r_{j+1}} |\nabla u(ry)|\,.$$

Hence, by Lemma 1 with $U = |\nabla u|, a = a_j = (r_j + r_{j+1})y/2$ and $\varepsilon = (r_{j+1} - r_j)/2 = 2^{-j-2}$,

$$|u(r_{j+1}y) - u(r_jy)|^p \le C2^{-jp}2^{jN} \int_{B(a_j,2^{-j-1})} |\nabla u(x)|^p\, dV(x)\,. \tag{6}$$

On the other hand, simple calculation shows that $|x-a_jy| \le 2^{-j-1}$ implies

$$2^{-j-2} \le 1 - |x|\,, \qquad |x-y| \le 2^{-j+1}\,.$$

Hence

$$2^{-j}2^{jN} \le 2^{N+2}P(x,y), \qquad \text{for } x \in B(a_j, 2^{-j-1})\,,$$

where P denotes the Poisson kernel,

$$P(x,y) = \frac{1-|x|^2}{|x-y|^N}\,. \tag{7}$$

From this and (6) we get

$$|u(r_{j+1}y) - u(r_jy)|^p \le C2^{-j(p-1)} \int_{r_{j-1}\le|x|\le r_{j+2}} P(x,y)|\nabla u(x)|^p\, dV(x)\,, \tag{8}$$

where we have used the inclusion

$$\{x : |x-a_j| \le 2^{-j-1}\} \subset \{x : r_{j-1} \le |x| \le r_{j+2}\,.\}$$

Now we integrate (8) over ∂B and use the formula

$$\int_S P(x,y)\, d\sigma(y) = 1$$

to get

$$\begin{aligned}\int_S |u(r_{j+1}y) - u(r_jy)|^p\, d\sigma(y) &\le C2^{-j(p-1)} \int_{r_{j-1}\le|x|\le r_{j+2}} |\nabla u(x)|^p\, dV(x)\\ &\le C\int_{r_{j-1}\le|x|\le r_{j+2}} (1-|x|)^{p-1}|\nabla u(x)|^p\, dV(x)\,.\end{aligned}$$

Combining this with (4) we get the desired result. □

Proof of Theorem 1. Let $n \ge 1$. By Lemma 2, we have

$$
\begin{aligned}
M_p^p(r_n,u) - |u(0)|^p &= M_p^p(r_n,u) - M_p^p(r_0,u) \\
&= \sum_{j=0}^{n-1} M_p^p(r_{j+1},u) - M_p^p(r_j,u) \\
&\leq C\sum_{j=0}^{n-1}\int_{r_{j-1}\leq|x|\leq r_{j+2}} (1-|x|)^{p-1}|\nabla u(x)|^p\,dV(x) \\
&\leq 3C\int_{|x|\leq r_{n+1}} (1-|x|)^{p-1}|\nabla u(x)|^p\,dV(x) \\
&\leq 3C\int_B (1-|x|)^{p-1}|\nabla u(x)|^p\,dV(x)\,.
\end{aligned}
$$

This proves the inequality

$$
M_p^p(r,u) \leq |u(0)|^p + C\int_B (1-|x|)^{p-1}|\nabla u(x)|^p\,dV(x)\,, \tag{9}
$$

for $r = r_n$. If $r \in (0,1)$ is arbitrary, we choose n so that $r_n \leq r \leq r_{n+1}$. Then we have

$$
|u(ry) - u(r_n y)| \leq 2^{-n} \sup_{r_n<r<r_{n+1}} |\nabla u(ry)|\,.
$$

Hence, by the proof of Lemma 2,

$$
\begin{aligned}
M_p^p(r,u) - M_p^p(r_n,u) &\leq C\int_{r_{n-1}\leq|x|\leq r_{n+2}} (1-|x|)^{p-1}|\nabla u(x)|^p\,dV(x) \\
&\leq C\int_B (1-|x|)^{p-1}|\nabla u(x)|^p\,dV(x)\,.
\end{aligned}
$$

This completes the proof.

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