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University of Niš

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APPLICATION OF CONVOLUTION AND DZIOK-SRIVASTAVA LINEAR OPERATORS ON ANALYTIC AND P-VALENT FUNCTIONS

Sh. Najafzadeh, S. R. Kulkarni and D. Kalaj

Abstract

In this paper, we introduce a new class of multivalent functions defined by convolution and Dziok-Srivastava operator and study some properties of this class e.g. coefficient estimates, integral representation, distortion and closure theorems, convolution and integral operator.

1 Introduction

Let $\mathcal{A}_p$ be the class of p-valent analytic functions with positive coefficients of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad z \in \Delta = \{z : |z| < 1\} \tag{1}$$

For two functions $f(z)$ given by (1) and

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \tag{2}$$

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the Hadamard product (or convolution) of $f(z)$ and $g(z)$ denoted by $(f * g)(z) = (g * f)(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k. \tag{3}$$

For $\{\alpha_1, \alpha_2, \cdots, \alpha_m\} \subseteq \mathbb{C}$ and $\{\beta_1, \beta_2, \cdots, \beta_n\} \subseteq \mathbb{C}.\}$ the generalized hypergeometric function ${}_mF_n(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n; z)$ is defined by

$$\begin{gathered} {}_mF_n(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_m)_k z^k}{(\beta_1)_k \cdots (\beta_n)_k k!} \\ (m \leq n+1, m, n, \in I\!N_0 = \{0, 1, 2, \cdots\}) \end{gathered} \tag{4}$$

where $(\lambda)_k$ is the pochhammer symbol defined by

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(k)} = \begin{cases} 1 & k = 0 \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & k \in I\!N \end{cases} \tag{5}$$

We consider Dziok - Srivastava operator [2] on $f(z) \in \mathcal{A}_p$ that is defined by

$$\begin{aligned} \mathcal{DS}_p^{m,n} &= \mathcal{DS}_p^{(m,n)}(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n) f(z) \\ &= h_p(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n; z) * f(z) \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_m)_{k-p} a_k z^k}{(\beta_1)_{k-p} \cdots (\beta_n)_{k-p} (k-p)!} \end{aligned} \tag{6}$$

where

$$h_p(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n; z) = z^p \; {}_mF_n(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n; z).$$

Definition : Let $g(z) = z^p + \sum\limits_{k=p+1}^{\infty} b_k z^k$ be a fixed p-valent analytic function in Δ. Define the class

$$\mathcal{A}_p(g(z), \alpha_1, \cdots, \alpha_m; \beta_1, \beta_2, \cdots, \beta_n, \gamma) = \mathcal{A}_p^{g(z)}(m, n, \gamma)$$

by

$$\begin{gathered} \mathcal{A}_p^{g(z)}(m, n, \gamma) = \{f(z) \in \mathcal{A}_p : Re\left\{1 + \frac{z(\mathcal{DS}_p^{m,n}(f*g)(z))''}{(\mathcal{DS}_p^{m,n}(f*g)(z))'}\right\} < p\gamma, \\ (1 < \gamma < 1 + \tfrac{1}{2p}, \quad z \in \Delta)\} \end{gathered} \tag{7}$$

For other subclasses of p-valent functions, we can see the recent works of authors in [1], [5], [7].

2 Main Results

In this section we first find a necessary and sufficient condition for functions to be in the class $\mathcal{A}_p^{g(z)}(m,n,\gamma)$.

Theorem 2.1 : $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$ if and only if

$$\sum_{k=p+1}^{\infty} \frac{k(k-p\gamma)}{p^2(\gamma-1)} \ \theta(k,p)\ a_k b_k \leq 1. \tag{8}$$

where

$$\theta(k,p) = \frac{(\alpha_1)_{k-p}\cdots(\alpha_m)_{k-p}}{(\beta_1)_{k-p}\cdots(\beta_n)_{k-p}(k-p)!}$$

Proof : If $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$, then by using (6) and (7) we obtain

$$Re\left\{1+\frac{z(p(p-1)z^{p-2}+\sum\limits_{k=p+1}^{\infty}\theta(k,,p)k(k-1)a_kb_kz^{k-2}}{pz^{p-1}+\sum\limits_{k=p+1}^{\infty}\theta(k,p)ka_kb_kz^{k-1}}\right\} < p\gamma$$

where

$$\theta(k,p) = \frac{(\alpha_1)_{k-p}\cdots(\alpha_m)_{k-p}}{(\beta_1)_{k-p}\cdots(\beta_n)_{k-p}(k-p)!}.$$

By letting $z \to 1^-$ through real values we have

$$\frac{p^2+\sum\limits_{k=p+1}^{\infty}\theta(k,p)k^2a_kb_k}{p+\sum\limits_{k=p+1}^{\infty}\theta(k,p)ka_kb_k} < p\gamma$$

or equivalently

$$\sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k,p)a_kb_k \leq p^2(\gamma-1).$$

To prove the "if" part, let (8) holds true, so

$$\left|\frac{z(\mathcal{DS}_p^{m,n}(f*g)(z))''-(p-1)(\mathcal{DS}_p^{m,n}(f*g)(z))'}{z(\mathcal{DS}_p^{m,n}(f*g)(z)''-[2p(1-\gamma)-1+p](\mathcal{DS}_p^{m,n}(f*g)(z))'}\right|$$

$$\leq \frac{\sum\limits_{k=p+1}^{\infty}k(k-p)a_kb_k}{2p^2(\gamma-1)-\sum\limits_{k=p+1}^{\infty}[k(k-p)(1-2(1-\gamma)))]a_kb_k} \leq 1,$$

so by maximum principal theorem the proof is complete. $\square$

Theorem 2.2 : If $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$, then

$$a_k \leq \frac{p^2(\gamma-1)}{k(k-p\gamma)b_k\theta(k,p)}. \tag{9}$$

the result is sharp for functions of the form

$$f_k(z) = z^p + \frac{p^2(\gamma-1)}{k(k-p\gamma)b_k\theta(k,p)}z^k \quad k = p+1, p+2, \cdots .$$

Proof : Let $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$. By (8), we have

$$k(k-p\gamma)\theta(k,p)a_kb_k \leq \sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k,p)a_kb_k \leq p^2(\gamma-1)$$

or

$$a_k \leq \frac{p^2(\gamma-1)}{k(k-p\gamma)\theta(k,p)b)k}$$

The sharpness is trivial and so omitted.

3 Distortion Bounds

In this section we obtain the distortion bounds for $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$.

Theorem 3.1 : If $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$, then

$$\begin{aligned} r^p - \frac{p^2(\gamma-1)}{(p+1)(p+1-p\gamma)\theta(p+1,p)b_{p+1}}r^{p+1} &\leq |f(z)| \\ \leq r^p + \frac{p^2(\gamma-1)}{(p+1)(p+1-p\gamma)\theta(p+1,p)b_{p+1}}r^{p+1} & \end{aligned} \tag{10}$$

where

$$\theta(p+1,p) = \frac{\prod_{i=1}^{m}\alpha_i}{\prod_{j=1}^{n}\beta_j}, \quad |z| = r < 1.$$

The result is sharp for function defined by

$$f(z) = z^p + \frac{p^2(\gamma-1)}{(p+1)(p+1-p\gamma)\theta(p+1,p)b_{p+1}}z^{p+1}. \tag{11}$$

Proof : By using (8), (9) we obtain

$$b_{p+1}\theta(p+1,p)(p+1)(p+1-p\gamma)\sum_{k=p+1}^{\infty}a_k \leq \sum_{k=p+1}^{\infty}k(k-p\gamma)\theta(k,p)a_kb_k \leq p^2(\gamma-1)$$

or

$$\sum_{k=p+1}^{\infty}a_k \leq \frac{p^2(\gamma-1)}{(p+1)(p+1-p\gamma)\theta(p+1,p)b_{p+1}} \tag{12}$$

For the function $f(z)=z^p+\sum\limits_{k=p+1}^{\infty}a_kz^k$ and using (12), for $|z|=r$ we have

$$\begin{aligned}|f(z)| &\leq r^p+\sum_{k=p+1}^{\infty}a_kr^k\\ &< r^p+r^{p+1}\sum_{k=p+1}^{\infty}a_k\\ &\leq r^p+\frac{p^2(\gamma-1)}{(p+1)(p+1-p\gamma)\theta(p+1,p)b_{p+1}}r^{p+1},\end{aligned}$$

also

$$\begin{aligned}|f(z)| &\geq r^p-\sum_{k=p+1}^{\infty}a_kr^k\\ &\geq r^p-\frac{p^2(\gamma-1)}{(p+1)(p+1-p\gamma)\theta(p+1,p)b_{p+1}}\ r^{p+1}.\end{aligned}$$

Now the proof is complete. □

Corollary : If $f(z)\in\mathcal{A}_p^{g(z)}(m,n,\gamma)$, then

$$\begin{aligned}&pr^{p-1}-\frac{p^2(\gamma-1)}{(p+1-p\gamma)\theta(p+1,p)b_{p+1}}r^p \leq |f'(z)|\\ &\leq pr^{p-1}+\frac{p^2(\gamma-1)}{(p+1-p\gamma)\theta(p+1,p)b_{p+1}}r^p.\end{aligned}$$

The result is sharp for the function given by (11).

4 Integral Representation

In this section we obtain integral representation for $\mathcal{DS}_p^{m,n}(f*g)(z)$ with suitable ranges on variables.

Theorem 4.1 : If $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$ then

$$\mathcal{DS}_p^{m,n}(f*g)(z) = \int_{\epsilon_2}^{z} \exp\left(\int_{\epsilon_1}^{z} \frac{p\gamma Q|t|-1}{t}dt\right) ds$$

where $\epsilon_1 \to 0, \epsilon_2 \to 0$ and $|Q(z)| < 1, z \in \Delta$.
Proof : By letting $\mathcal{DS}_p^{m,n}(f*g)(z) = M(z)$ in (7) we have

$$Re\left\{1 + \frac{zM''(z)}{M'}\right\} < p\gamma.$$

Since for all z,

$$\text{Re}\left\{1 + \frac{zM''}{M'}\right\} < \left|1 + \frac{zM''}{M'}\right\},$$

hence by choosing the values of z such that

$$\left|1 + \frac{zM'}{M'}\right| < p\gamma$$

we conclude

$$1 + \frac{zM''}{M'} = p\gamma Q(z) \quad \text{or} \quad \frac{M''}{M'} = \frac{p\gamma Q(z) - 1}{z}.$$

After integration we obtain

$$\log(M'(z)) = \int_{\epsilon_1}^{z} \frac{p\gamma Q(t) - 1}{t}dt \quad (\epsilon_1 \to 0),$$

thus

$$M'(z) = \exp\int_{\epsilon_1}^{z} \frac{p\gamma Q(t) - 1}{t}dt \quad (\epsilon_1 \to 0),$$

after integration we obtain the result. □

5 Some Properties of the Class $\mathcal{A}_p^{g(z)}(m,n,\gamma)$

In this section, we prove the closure theorems for the class $\mathcal{A}_p^{g(z)}(m,n,\gamma)$.
Theorem 5.1 : Let $F_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j}z^k \quad (j = 1,2,\cdots,q)$ be in the class $\mathcal{A}_p^{g(z)}(m,n,\gamma)$ and $\eta_j \geq 0$ for $j = 1,2,\cdots,q$ and $\sum_{j=1}^{q} \eta_j \leq 1$ then the function

$$f(z) = z^p + \sum_{k=p+1}^{\infty}\left(\sum_{j=1}^{q} \eta_j a_{k,j}\right) z^k$$

belongs to $\mathcal{A}_p^{g(z)}(m,n,\gamma)$.

Proof : Since $F_j(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$, then from Theorem 2.1 for every $j = 1, 2, \cdots, q$ we have

$$\sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k,p)b_k a_{k,j} \leq p^2(\gamma-1)$$

Also

$$\begin{aligned}&\sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k,p)b_k \left(\sum_{j=1}^{q} \eta_j a_{k,j}\right)\\ &= \sum_{j=1}^{q} \eta_j (\sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k,p)b_k a_{k,j})\\ &\leq \sum_{j=1}^{q} \eta_j p^2(\gamma-1)\\ &\leq p^2(\gamma-1).\end{aligned}$$

So by Theorem 2.1, $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$.

Theorem 5.2 : Let $F_p(z) = z^p$ and

$$F_k(z) = z^p + \frac{p^2(\gamma-1)}{k(k-p\gamma)\theta(k,p)b_k} z^k, \quad (k = p+1, \cdots).$$

Then $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$ if and only if

$$f(z) = \eta_p z^p + \sum_{k=p+1}^{\infty} \eta_k F_k(z)$$

where $\sum\limits_{k=p}^{\infty} \eta_k = 1$ and $\eta_k \geq 0$.

Proof : Let $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$, then from Theorem 2.2, we have

$$a_k \leq \frac{p^2(\gamma-1)}{k(k-p\gamma)\theta(k,p)b_k} \quad (k = p+1, p+2, \cdots)$$

therefore by letting

$$\eta_k = \frac{k(k-p\gamma)\theta(k,p)b_k a_k}{p^2(\gamma-1)} \quad (k = p+1, p+2, \cdots)$$

and $\eta_p = 1 - \sum_{k=p+1}^{\infty} \eta_k$. We conclude the required result. Conversely, let $f(z) = \eta_p z^p + \sum_{k=p+1}^{\infty} \eta_k F_k(z)$, then

$$\begin{aligned} f(z) &= \eta_p z^p + \sum_{k=p+1}^{\infty} \eta_k \left(z^p + \frac{p^2(\gamma-1)}{k(k-p\gamma)\theta(k,p)b_k} z^k \right) \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{\eta_k p^2(\gamma-1)}{k(k-p\gamma)\theta(k,p)b_k} z^k. \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{k=p+1}^{\infty} \frac{\eta_k p^2(\gamma-1)}{k(k-p\gamma)\theta(k,p)b_k} \;\; \frac{k(k-p\gamma)}{p^2(\gamma-1)}\theta(k,p)b_k \\ &= \sum_{k=p+1}^{\infty} \eta_k = 1 - \eta_p \leq 1. \end{aligned}$$

Hence by Theorem 2.1, we have $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$.

6 Convolution Property and Integral Operator

In this section we show that the class $\mathcal{A}_p^{g(z)}(m,n,\gamma)$ is closed under convolution and one special operator.

Theorem 6.1 : Let $h(z) = z^p + \sum_{k=p+1}^{\infty} c_k z^k$ be analytic in unit disk Δ and $0 \leq c_k \leq 1$. If $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$, then $(f * h)(z)$ is also in the class $\mathcal{A}_p^{g(z)}(m,n,\gamma)$.

Proof : Since $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$ then by Theorem 2.1 we have

$$\sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k,p)a_k b_k \leq p^2(\gamma-1)$$

By using the last inequality and the fact that

$$(f*h)(z) = z^p + \sum_{k=p+1}^{\infty} a_k c_k z^k$$

we have

$$\sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k,p)a_k c_k b_k$$
$$\leq \sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k,p)a_k b_k \leq p^2(\gamma-1)$$

and this by Theorem 2.1 gives the result. □

Theorem 6.2 : If $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$, then

$$F(z) = \frac{\lambda+p}{z^{\lambda}} \int_0^z t^{\lambda-1} f(t)dt \quad (\lambda > -1; \quad z \in \Delta)$$

is also in the class $\mathcal{A}_p^{g(z)}(m,n,\gamma)$. See [3], [5].

Proof : Since $F(z) = f(z) * \left(z^p + \sum\limits_{k=p+1}^{\infty} \frac{\lambda+p}{\lambda+k} z^k \right)$ and $\frac{\lambda+p}{\lambda+k} \leq 1$, by Theorem 6.1, the proof is trivial.

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Shahram Najafzadeh
West Azarbayjan, Orumiyeh, Emamat Bolvar, University Of Applied Science & Technology P.O.Box : 1114, Iran
E-mail: Najafzadeh1234@yahoo.ie

S. R. Kulkarni
Department of Mathematics, Fergusson College, Pune University, Pune - 411004, India
E-mail: kulkarni_ferg@yahoo.com

David Kalaj
University of Montenegro, Faculty of Natural Sciences and Mathematics, Cetinjski put b.b. 8100 Podgorica, Montenegro
E-mail: davidk@cg.yu