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TOTALLY UMBILICAL SEMI-INVARIANT SUBMANIFOLDS OF A NEARLY COSYMPLECTIC MANIFOLD

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Abstract. A classification theorem for a totally umbilical semi-invariant submanifold of a nearly cosymplectic manifold is proved.

1. Introduction

The study of semi-invariant submanifold or contact CR-submanifolds of almost contact metric manifold was initiated by A. Bejancu and N. Papaghiuc [1] and was followed up by several other geometers (c.f., [3], [5], [6],). In particular, semi-invariant submanifolds of different classes of almost contact metric manifolds have also been studied (c.f., [7], [8],). In the present note we study semi-invariant submanifolds of a nearly cosymplectic manifolds and have worked out a classification for totally umbilical semi-invariant submanifolds of a nearly cosymplectic manifold.

Let $\bar{M}$ be an almost contact metric manifold with almost contact metric structure (ϕ, ξ, η, g), that is ϕ is a (1, 1) tensor field, ξ is a vectorfield, η is a 1-form and g is the compatible Riemannian metric. Such that

$$(1.1) \quad \begin{cases} \phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), . \end{cases}$$

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for each $X, Y \in T\bar{M}$ where $T\bar{M}$ denotes the tangent bundle of $\bar{M}$.

An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is a nearly cosymplectic structure [4] if

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = 0. \tag{1.2}$$

A cosymplectic structure is always a nearly cosymplectic structure.

An m-dimensional submanifold M of $\bar{M}$ is said to be a semi-invariant submanifold if there exist a pair of orthogonal distributions $(D, D^{\perp})$ satisfying the conditions

(i) $TM = D \oplus D^{\perp} \oplus \langle \xi \rangle$.
(ii) The distribution D is invariant by ϕ i.e., $\phi D_x = D_x \;\; \forall\, x \in M$.
(iii) The distribution $D^{\perp}$ is anti-invariant i.e., $\phi D_x^{\perp} \subseteq T_x^{\perp} M$, $\forall\, x \in M$.

Where $\langle \xi \rangle$ is the distribution spanned by the structure vector field ξ.

Let TM denote the tangent bundle on M. The orthogonal complement of $\phi D^{\perp}$ in the normal bundle $T^{\perp}M$ is an invariant subbundle of $T^{\perp}M$ under ϕ and is denoted by μ i.e.,

$$T^{\perp}M = \phi D^{\perp} \oplus \mu.$$

For $U, V \in TM$ and $N \in T^{\perp}M$, the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_U V = \nabla_U V + h(U, V)$$

$$\bar{\nabla}_U N = -A_N U + \nabla_U^{\perp} N.$$

Where ∇ and $\nabla^{\perp}$ are symbols used for connection on TM and $T^{\perp}M$ respectively. While h and A_N denote the second fundamental forms related by $g(h(U, V), N) = g(A_N U, V)$ and g is the Riemannian metric on $\bar{M}$ as well as on M.

The transformation ϕU and ϕN are decomposed into tangential and normal parts respectively as

$$\phi U = PU + FU \tag{1.3}$$

$$\phi N = tN + fN \tag{1.4}$$

Note 1.1. It is easy to observe that $PU \in D, FU \in \phi D^{\perp}, tN \in D^{\perp}$ and $fN \in \mu$.

Now, denoting by $P_U V$ and $Q_U V$ the tangential and normal parts of $(\bar{\nabla}_U \phi)V$ and making use of equations (1.4), (1.5), the Gauss and Weingarten formulae, the following equations may easily be obtained

$$(1.5)\quad \begin{cases} P_U V = (\nabla_U P)V - A_{FV}U - th(U,V) \\ Q_U V = (\nabla_U F)V + h(U,PV) - fh(U,V). \end{cases}$$

Where the covariant derivatives of P and F are defined by

$$(\nabla_U P)V = \nabla_U PV - P\nabla_U V$$

$$(\nabla_U F)V = \nabla^{\perp}_U FV - F\nabla_U V.$$

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be totally umbilical submanifold if the second fundamental form satisfies

$$h(U,V) = g(U,V)H$$

where H is the mean curvature vector.

2. Semi-invariant submanifolds of a nearly cosymplectic manifold

To develop the proof of main theorem, we start with the following preparatory results

Proposition 2.1. Let M be a semi-invariant submanifold of a nearly cosymplectic manifold $\bar{M}$ with $h(X,\phi X) = 0$ for each $X \in D \oplus \langle\xi\rangle$. If the invariant distribution $D \oplus \langle\xi\rangle$ is integrable, then each of its leaves is totally geodesic in M as well as in $\bar{M}$.

Proof. For $X, Y \in D \oplus \langle\xi\rangle$, we have

$$h(X,\phi Y) + h(\phi X, Y) = (\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X + \phi(\bar{\nabla}_X Y + \bar{\nabla}_Y X)$$
$$-(\nabla_X\phi Y + \nabla_Y\phi X)$$

Taking account of the hypothesis and the formula (1.2), the above equation takes the form

$$(2.1)\qquad 0 = \phi(\nabla_X Y + \nabla_Y X) + 2\phi h(X,Y) - (\nabla_X\phi Y + \nabla_Y\phi X).$$

On equating the normal parts in the right hand side of the last equation to zero, we get

$$2fh(X,Y) = F(\nabla_X Y + \nabla_Y X),$$

from which it follows that

$$(2.2)\qquad \nabla_X Y + \nabla_Y X \in D \oplus \langle\xi\rangle$$

and

$$(2.3)\qquad h(X,Y) \in \phi D^{\perp}.$$

As $D \oplus \langle\xi\rangle$ is integrable, it follows from the observation (2.2) that

$$(2.4)\qquad \nabla_X Y \in D \oplus \langle\xi\rangle.$$

Taking account of this fact in equation (2.1), it follows that

$$h(X, Y) = 0 \tag{2.5}$$

The assertion is proved by virtue of (2.4) and (2.5).

The above proposition leads to the following consequence which is in itself an important result with geometric point of view.

Corollary 2.1. Let M be a totally umbilical semi-invariant submanifold of a nearly cosymplectic manifold $\bar{M}$. If the invariant distribution $D \oplus \langle\xi\rangle$ on M is integrable, then M is totally geodesic in $\bar{M}$.

To workout an integrability condition for the anti-invariant distribution $D^{\perp} \oplus \langle\xi\rangle$, we take vector field $Z, W \in D^{\perp} \oplus \langle\xi\rangle$ and $U \in TM$ and write

$$\begin{aligned}
2g(A_{\phi Z}W, U) &= g(h(U, W), \phi Z) + g(h(U, W), \phi Z) \\
&= g(\bar{\nabla}_W U, \phi Z) + g(\bar{\nabla}_U W, \phi Z) \\
&= -g(\bar{\nabla}_W \phi U + \bar{\nabla}_U \phi W, Z) + g((\bar{\nabla}_U \phi)W \\
&\quad + (_W\phi)U, Z) \\
&= g(\phi U, \bar{\nabla}_W Z) + g(A_{\phi W}U, Z) \\
&= -g(U, \phi\bar{\nabla}_W Z) + g(A_{\phi W}U, Z).
\end{aligned}$$

As U is an arbitrary vector field on M, we obtain

$$2A_{\phi Z}W = A_{\phi W}Z - \phi\bar{\nabla}_W Z.$$

Similarly,

$$2A_{\phi W}Z = A_{\phi Z}W - \phi\bar{\nabla}_Z W.$$

On making substraction, we get

$$3(A_{\phi Z}W - A_{\phi W}Z) = \phi[Z, W]$$

which on operating ϕ and using equation (1.1) gives

$$[Z, W] = \eta([Z, W])\xi + 3\phi(A_{\phi W}Z - A_{\phi Z}W) \tag{2.6}$$

equation (2.6) leads to the following

Proposition 2.2. Let M be a semi-invariant submanifold of a nearly cosymplectic manifold $\bar{M}$, then the distribution $D^{\perp} \oplus \langle\xi\rangle$ is integrable if and only if

$$A_{\phi Z}W = A_{\phi W}Z$$

for each $Z, W \in D^{\perp} \oplus \langle\xi\rangle$.

Corollary 2.2. On a semi-invariant submanifold of a cosymplectic manifold the distribution $D^{\perp} \oplus \langle \xi \rangle$ is integrable.

3. Totally umbilical semi-invariant submanifolds of a nearly cosymplectic manifold

Throughout this section M denotes a totally umbilical semi-invariant submanifold of a nearly cosymplectic manifold $\bar{M}$. For $U \in TM$ by formula (1.2)

$$(\bar{\nabla}_U \phi)U = 0.$$

In particular for $Z \in D^{\perp}$,

$$(\bar{\nabla}_Z \phi)Z = 0$$

and therefore,

$$P_Z Z = 0 \tag{3.1}$$

and

$$Q_Z Z = 0.$$

On applying first equation of (1.5) and using the fact that $PZ = 0$, equation (3.1) yields

$$-g(H, FZ)Z - \|Z\|^2 tH = P\nabla_Z Z.$$

In view of Note 1.1, we have

$$g(H, FZ)Z + \|Z\|^2 tH = 0 \tag{3.2}$$

equation (3.2) has solutions if either

(a) dim $D^{\perp} = 1$, (b) $H \in \mu$ or (c) $D^{\perp} = \{0\}$.

Now, we are in position to prove the main theorem

Theorem 3.1. Let M be a totally umbilical semi-invariant submanifold of a nearly cosymplectic manifold $\bar{M}$. Then at least one of the following is true

(i) M is anti-invariant.
(ii) M is totally geodesic.
(iii) dim $(D^{\perp}) = 1$ and $D \oplus \langle \xi \rangle$ is not integrable

Proof. If $D = \{0\}$, then by definition M is anti-invariant which is case i. If $D \neq \{0\}$ and $D \oplus \langle \xi \rangle$ is integrable then by Corollary 2.1, M is totally geodesic in $\bar{M}$ which account for case ii. If $D \oplus \langle \xi \rangle$ is not integrable and $H \in \mu$ then by virtue of observation (2.3), M is again totally geodesic. If however $H \notin \mu$, then equation (3.2) has solutions if and only if dim $D^{\perp} = 1$ or $D^{\perp} = \{0\}$. For the case when dim $D^{\perp} = 1$, M belongs to the category (iii). Lastly, if $H \notin \mu$ and $D^{\perp} = \{0\}$. Then again by Corollary 2.1, M is totally geodesic. This completes the proof.

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