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THE STRENGTHENED HARDY INEQUALITIES AND THEIR NEW GENERALIZATIONS

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Abstract. In this article, using the properties of power mean, new generalizations of the strengthened Hardy Inequalities are proved.

1. Introduction

It is well known that the following Hardy's Inequality (see [4, Theorem 326]):

if $p > 1$ and $a_n \geq 0$, then

$$\sum\Big(\frac{a_1 + a_2 + ... + a_n}{n}\Big)^p < \Big(\frac{p}{p-1}\Big)^p \sum a_n^p, \tag{1.1}$$

unless all the a are zero. The constant is the best possible.

This theorem was discovered in the course of attempts to simplify the proofs then known of Hilbert's double series theorems (see [4, Theorem 315]). Hilbert's double series theorem was completed by the above inequality. This

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inequality was first proved by Hardy [3], except that Hardy was unable to fit the constant in inequality (1.1). If in inequality (1.1) we write a_n for a_n^p, we obtain

$$\sum\Big(\frac{a_1^{1/p}+a_2^{1/p}+\cdots+a_n^{1/p}}{n}\Big)^p<\Big(\frac{p}{p-1}\Big)^p\sum a_n. \tag{1.2}$$

If we make $p\to\infty$, and use the elementary mean values

$$\lim_{p\to 0}\Big(\sum_{i=1}^{n}\frac{1}{n}a_i^p\Big)^{1/p}=\Big(\prod_{i=1}^{n}a_i\Big)^{1/n},$$

we obtain

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n}<e\sum_{n=1}^{\infty}a_n,$$

and this suggests the more complete theorem which follow;

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n}<e\sum_{n=1}^{\infty}a_n, \tag{1.3}$$

unless (a_n) *is null. The constant is the best possible.*

The inequality given in (1.3) which later went by the name of *Carleman's inequality,* led to a great many papers dealing with alternative proofs, various generalizations, and numerous variants and applications in analysis. It is natural to attempt to prove the complete inequality by means of following

$$\Big(\prod_{i=1}^{n}a_i\Big)^{1/n}<\sum_{i=1}^{n}\frac{1}{n}a_i, \tag{1.4}$$

unless all the a_i are equal. But a direct application of inequality (1.4) to the left-hand side of the inequality (1.2) is insufficient. To remedy this, we apply inequality (1.4) not to $a_1,a_2,...,a_n$ but to $c_1a_1,c_2a_2,...,c_na_n$, and choose the c so that when $\sum a_n$ is near the boundary of convergence, these numbers shall be 'roughly equal'. This requires that c_n shall be roughly of the order of n.

By Hardy (see, [4, Theorem 349]), the Carleman's inequality was generalized as follows:

If $a_n\geq 0,\lambda_n>0$, $\Lambda_n=\sum_{m=1}^{n}\lambda_m (n\in N)$ *and* $0<\sum_{n=1}^{\infty}\lambda_na_n<\infty$, *then*

$$\sum_{n=1}^{\infty}\lambda_n(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/\Lambda_n}<e\sum_{n=1}^{\infty}\lambda_na_n. \tag{1.5}$$

Recently, Z. Xie and Y. Zhong [7] gave an improvement of the inequality (1.5) as follows: *If* $a_n \geq 0, 0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m (n \in N)$ *and* $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, *then*

$$\sum_{n=1}^{\infty} \lambda_{n+1}(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \Big(1 - \frac{6\lambda_n}{12\Lambda_n + 11\lambda_n}\Big) \lambda_n a_n. \tag{1.6}$$

Most recently, Z. Yang [11] obtained the strengthened Carleman's inequality as follows: *If* $a_n \geq 0$, $n = 1, 2, ...,$ *and* $0 < \sum_{n=1}^{\infty} a_n < \infty$. *Then*

$$\begin{aligned} &\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \\ &< e \sum_{n=1}^{\infty} \Big(1 - \frac{1}{2(1+n)} - \frac{1}{24(1+n)^2} - \frac{1}{48(1+n)^3}\Big) a_n. \end{aligned} \tag{1.7}$$

It is immediate from the proof of inequality (1.6) and the inequality (1.7) that we can deduce the following new strengthened Hardy's inequality:

$$\begin{aligned} &\sum_{n=1}^{\infty} \lambda_{n+1}(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ &< e \sum_{n=1}^{\infty} \Big(1 - \frac{\lambda_n}{2(\Lambda_n + \lambda_n)} - \frac{\lambda_n^2}{24(\Lambda_n + \lambda_n)^2} - \frac{\lambda_n^3}{48(\Lambda_n + \lambda_n)^3}\Big) \lambda_n a_n. \end{aligned} \tag{1.8}$$

But we know that the inequality (1.8) is a better improvement of the inequality (1.6), as a result of following

$$\Big(1 - \frac{\lambda_n}{2(\Lambda_n + \lambda_n)} - \frac{\lambda_n^2}{24(\Lambda_n + \lambda_n)^2} - \frac{\lambda_n^3}{48(\Lambda_n + \lambda_n)^3}\Big) < \Big(1 - \frac{6\lambda_n}{12\Lambda_n + 11\lambda_n}\Big)$$

for $\Lambda_n / \lambda_n \geq 1$.

The purpose of this paper is to prove new extension of the strengthened Hardy's inequality in the spirit of the strict monotonicity of the power mean of n distinct positive numbers.

For any positive values $a_1, a_2, \ldots, a_n$ and positive weights $\alpha_1, \alpha_2 \ldots, \alpha_n$, $\sum_{i=1}^{n} \alpha_i = 1$, and any real $p \neq 0$, we defined the power mean, or the mean of order p of the value a with weights α by

$$M_p(a;\alpha) = M_p(a_1, a_2, \ldots, a_n; \alpha_1, \alpha_2 \ldots, \alpha_n) = \Big(\sum_{i=1}^{n} \alpha_i a_i^p\Big)^{1/p}.$$

An easy application of L'Hospital's rule shows that

$$\lim_{p\to 0} M_p(a;\alpha) = \prod_{i=1}^{n} a_i^{\alpha_i},$$

the geometric mean. Accordingly, we define $M_0(a;\alpha) = \prod_{i=1}^{n} a_i^{\alpha_i}$. It is well known that $M_p(a;\alpha)$ is a nondecreasing function of p for $-\infty \leq p \leq \infty$, and is strictly increasing unless all the a_i are equal (cf. [1]).

2. Strengthened Hardy's Inequalities

The main results of this paper are presented as follows:

Lemma 2.1 [7]. *Let $x \geq 1$, then we have the following inequality:*

$$\frac{12x+11}{12x+5}\Big(1+\frac{1}{x}\Big)^x < e < \frac{14x+12}{14x+5}\Big(1+\frac{1}{x}\Big)^x. \tag{2.1}$$

We can deduce the following improvement results of the inequality (1.6):

Theorem 2.2. *Let $0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m (\Lambda_n \geq 1)$, $a_n \geq 0 (n \in N)$, $0 < p \leq 1$ and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$. Then*

$$\begin{aligned} &\sum_{n=1}^{\infty} \lambda_{n+1}(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ &< \frac{e^p}{p} \sum_{n=1}^{\infty} \Big(1 - \frac{6\lambda_n}{12\Lambda_n + 11\lambda_n}\Big)^p \lambda_n (a_n)^p \Lambda_n^{p-1} \Big(\sum_{k=1}^{n} \lambda_k (c_k a_k)^p\Big)^{(1-p)/p}. \end{aligned} \tag{2.2}$$

where $c_k^{\lambda_n} = (\Lambda_{n+1})^{\Lambda_n} / (\Lambda_n)^{\Lambda_{n-1}}$.

Proof. By the power mean inequality, we have

$$\alpha_1^{q_1} \alpha_2^{q_2} \cdots \alpha_n^{q_n} \leq \Big(\sum_{m=1}^{n} q_m (\alpha_m)^p\Big)^{1/p},$$

for $\alpha_m \geq 0$, $p \geq 0$ and $q_m > 0 (m = 1, 2, \ldots, n)$ with $\sum_{m=1}^{n} q_m = 1$. Setting $c_m > 0, \alpha_m = c_m a_m$ and $q_m = \lambda_m/\Lambda_n$, we obtain

$$(c_1a_1)^{\lambda_1/\Lambda_n}(c_2a_2)^{\lambda_2/\Lambda_n}\cdots(c_na_n)^{\lambda_n/\Lambda_n} \leq \Big(\frac{1}{\Lambda_n}\sum_{m=1}^{n}\lambda_m(c_ma_m)^p\Big)^{1/p}.$$

Using the above inequality, we have

$$\begin{aligned} &\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ &= \sum_{n=1}^{\infty}\lambda_{n+1}\frac{(c_1a_1)^{\lambda_1/\Lambda_n}(c_2a_2)^{\lambda_2/\Lambda_n}\cdots(c_na_n)^{\lambda_n/\Lambda_n}}{(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{1/\Lambda_n}} \\ &\leq \sum_{n=1}^{\infty}\Big[\frac{\lambda_{n+1}}{(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{1/\Lambda_n}}\Big]\Big(\frac{1}{\Lambda_n}\sum_{m=1}^{n}\lambda_m(c_ma_m)^p\Big)^{1/p}. \end{aligned} \tag{2,3}$$

By using the following inequality (see [2], [6]),

$$\Big(\sum_{m=1}^{n} z_m\Big)^t \leq t\sum_{m=1}^{n} z_m\Big(\sum_{k=1}^{m} z_k\Big)^{t-1},$$

where $t \geq 1$ is constant and $z_m \geq 0 (m = 1, 2, \cdots)$, it is easy to observe that

$$\begin{aligned} &\Big(\frac{1}{\Lambda_n}\sum_{m=1}^{n}\lambda_m(c_ma_m)^p\Big)^{1/p} \\ &\leq \frac{1}{\Lambda_n}\Big(\sum_{m=1}^{n}\lambda_m(c_ma_m)^p\Big)^{1/p} \\ &\leq \frac{1}{p\Lambda_n}\sum_{m=1}^{n}\lambda_m(c_ma_m)^p\Big(\sum_{k=1}^{m}\lambda_k(c_ka_k)^p\Big)^{(1-p)/p} \end{aligned} \tag{2.4}$$

for $\Lambda_n \geq 1$ and $0 < p \leq 1$. Then, by (2.3) and (2.4), we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ &\leq \frac{1}{p}\sum_{m=1}^{n}\lambda_m(c_ma_m)^p\sum_{n=m}^{\infty}\Big(\frac{\lambda_{n+1}}{\Lambda_n(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{1/\Lambda_n}}\Big)\Big(\sum_{k=1}^{m}\lambda_k(c_ka_k)^p\Big)^{(1-p)/p} \end{aligned}$$

Choosing $c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n} = (\Lambda_{n+1})^{\Lambda_n} \quad (n \in N)$ and setting $\Lambda_0 = 0$, from $\lambda_{n+1} \le \lambda_n$, it follows that

$$\begin{aligned} c_n &= \Big[\frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}\Big]^{1/\lambda_n} = \Big(1+\frac{\lambda_{n+1}}{\Lambda_n}\Big)^{\Lambda_n/\lambda_n}\cdot\Lambda_n \\ &\le \Big(1+\frac{\lambda_n}{\Lambda_n}\Big)^{\Lambda_n/\lambda_n}\cdot\Lambda_n. \end{aligned}$$

This implies that

$$\begin{aligned} &\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ &\le \frac{1}{p}\sum_{m=1}^{n}\lambda_m(c_m a_m)^p\sum_{n=m}^{\infty}\frac{\lambda_{n+1}}{\Lambda_n\Lambda_{n+1}}\Big(\sum_{k=1}^{m}\lambda_k(c_k a_k)^p\Big)^{(1-p)/p} \\ &= \frac{1}{p}\sum_{m=1}^{n}\lambda_m(c_m a_m)^p\sum_{n=m}^{\infty}\Big(\frac{1}{\Lambda_n}-\frac{1}{\Lambda_{n+1}}\Big)\Big(\sum_{k=1}^{m}\lambda_k(c_k a_k)^p\Big)^{(1-p)/p} \\ &= \frac{1}{p}\sum_{m=1}^{n}\lambda_m(c_m a_m)^p\frac{1}{\Lambda_m}\Big(\sum_{k=1}^{m}\lambda_k(c_k a_k)^p\Big)^{(1-p)/p} \\ &\le \frac{1}{p}\sum_{m=1}^{\infty}\Big(1+\frac{1}{\Lambda_m/\lambda_m}\Big)^{p\Lambda_m/\lambda_m}\lambda_m(a_m)^p\Lambda_m^{p-1}\Big(\sum_{k=1}^{m}\lambda_k(c_k a_k)^p\Big)^{(1-p)/p}. \end{aligned}$$

Hence, by the above inequality and Lemma 2.1, we have

$$\begin{aligned} &\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ &< \frac{e^p}{p}\sum_{n=1}^{\infty}\Big(1-\frac{6\lambda_n}{12\Lambda_n+11\lambda_n}\Big)^p\lambda_n(a_n)^p\Lambda_n^{p-1}\Big(\sum_{k=1}^{n}\lambda_k(c_k a_k)^p\Big)^{(1-p)/p}. \end{aligned}$$

Thus Theorem 2.2 is proved. □

Setting $p \equiv 1$ in Theorem 2.2, then, form inequality (2.2) we have the inequality (1.6). Also assuming that $\lambda_n = 1$ in the Theorem, we have an extension of the strengthened Carleman's inequality as following:

Corollary 2.3. *Let $a_n \ge 0 (n \in N)$, $0 < p \le 1$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$. Then*

$$\begin{aligned} &\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} \\ &< \frac{e^p}{p}\sum_{n=1}^{\infty}\Big(1-\frac{6}{12n+11}\Big)^p(a_n)^p n^{p-1}\Big(\sum_{k=1}^{n}(c_k a_k)^p\Big)^{(1-p)/p}. \end{aligned}$$

where $c_k = (1 + 1/k)^k \cdot k$.

Similarly to Theorem 2.2, we can consider a generalization version of the inequality (1.8) as following theorem:

Theorem 2.4. *Let* $0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$, $a_n \geq 0 (n \in N)$, $0 < p \leq 1$ *and* $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$. *Then*

$$\begin{aligned} &\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ &< \frac{e}{p} \sum_{n=1}^{\infty} \Big(1 - \frac{\lambda_n}{2(\Lambda_n + \lambda_n)} - \frac{\lambda_n^2}{24(\Lambda_n + \lambda_n)^2} - \frac{\lambda_n^3}{48(\Lambda_n + \lambda_n)^3}\Big)^p \\ &\quad \times \lambda_n (a_n)^p \Lambda_n^{p-1} \Big(\sum_{k=1}^{n} \lambda_k (c_k a_k)^p\Big)^{(1-p)/p}. \end{aligned} \tag{2.5}$$

The proof is almost the same as in proving Theorem 2.2. We here only need to note that

$$\Big(1 + \frac{1}{x}\Big)^x < e\Big(1 - \frac{1}{2(1+x)} - \frac{1}{2(1+x)^2} - \frac{1}{2(1+x)^3}\Big)$$

for $x > 0$, which proved in [11, Lemma 1].

Corollary 2.5. *Let* $a_n \geq 0 (n \in N)$, $0 < p \leq 1$ *and* $0 < \sum_{n=1}^{\infty} a_n < \infty$. *Then*

$$\begin{aligned} &\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \\ &< \frac{e^p}{p} \sum_{n=1}^{\infty} \Big(1 - \frac{1}{2(1+n)} - \frac{1}{24(1+n)^2} - \frac{1}{48(1+n)^3}\Big)^p \\ &\quad \times (a_n)^p n^{p-1} \Big(\sum_{k=1}^{n} (c_k a_k)^p\Big)^{(1-p)/p}. \end{aligned}$$

where $c_k = (1 + 1/k)^k \cdot k$.

Lemma 2.6. *If $a_1, a_2, \dots, a_n > 0$ and $\alpha_1, \alpha_2 \dots, \alpha_n > 0$ with $\sum_{i=1}^{n} \alpha_i = 1$, then we have the following inequality:*

$$\Big(\prod_{i=1}^{n} a_i^{\alpha_i}\Big)^k \le \Big(\sum_{i=1}^{n} \alpha_i (a_i)^p\Big)^{k/p}$$

for $0 < k, p$ with the equality holding if and only if all a_i are same.

Note that Lemma 2.6 is easily deduced form the fact that $M_p(a;\alpha)$ is a continuous strictly increasing function of p.

Now, we are ready to introduce the following new general strengthened Hardy's inequality.

Theorem 2.7. *Let $0 < \lambda_{n+1} \le \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m (\Lambda_n \ge 1)$, $a_n \ge 0 (n \in N)$ and $0 < \sum_{n=1}^{\infty} \lambda_n (a_n)^t < \infty$ for $0 < p \le t < \infty$. Then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n} < \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \Big(1 - \frac{6\lambda_n}{12\Lambda_n + 11\lambda_n}\Big)^{p/t}$$
$$\times \lambda_n (a_n)^p \Lambda_n^{(p-t)/t} \Big(\sum_{k=1}^{n} \lambda_k c_k a_k\Big)^{(t-p)/p}. \tag{2.6}$$

Proof. The proof is almost the same as in Theorem 2.2. By Lemma 2.6, we have

$$(\alpha_1^{q_1} \alpha_2^{q_2} \cdots \alpha_n^{q_n})^t \le \Big(\sum_{m=1}^{n} q_m (\alpha_m)^p\Big)^{t/p}, \quad p, t \ge 0,$$

where $\alpha_m \ge 0$ and $q_m > 0 (m = 1, 2, \dots, n)$ with $\sum_{m=1}^{n} q_m = 1$. Setting $c_m > 0, \alpha_m = c_m a_m$ and $q_m = \lambda_m / \Lambda_n$, we obtain

$$\big((c_1 a_1)^{\lambda_1/\Lambda_n} (c_2 a_2)^{\lambda_2/\Lambda_n} \cdots (c_n a_n)^{\lambda_n/\Lambda_n}\big)^t \le \Big(\frac{1}{\Lambda_n} \sum_{m=1}^{n} \lambda_m (c_m a_m)^p\Big)^{t/p}.$$

Using the above inequality, we have

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n}$$
$$\le \sum_{n=1}^{\infty} \Big[\frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{t/\Lambda_n}}\Big] \frac{1}{\Lambda_n} \Big(\sum_{m=1}^{n} \lambda_m (c_m a_m)^p\Big)^{t/p} \tag{2.7}$$

for $\Lambda_n \geq 1$ and $t \geq p$. By using the following inequality (see [2], [6]),

$$\Big(\sum_{m=1}^{n} z_m\Big)^t \leq t\sum_{m=1}^{n} z_m\Big(\sum_{k=1}^{m} z_k\Big)^{t-1},$$

where $t \geq 1$ is constant and $z_m \geq 0 (m = 1, 2, \cdots)$, it is easy to observe that

$$\Big(\sum_{m=1}^{n}\lambda_m(c_m a_m)^p\Big)^{t/p} \leq \frac{t}{p}\sum_{m=1}^{n}\lambda_m(c_m a_m)^p\Big(\sum_{k=1}^{m}\lambda_k(c_k a_k)^p\Big)^{(t-p)/p}. \tag{2.8}$$

for $\Lambda_n \geq 1$ and $t \geq p$. Then, by (2.7) and (2.8), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{t/\Lambda_n} &\leq \sum_{n=1}^{\infty}\Big[\frac{\lambda_{n+1}}{(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{t/\Lambda_n}}\Big]\frac{1}{\Lambda_n}\frac{t}{p}\\ &\times\sum_{m=1}^{n}\lambda_m(c_m a_m)^p\Big(\sum_{k=1}^{m}\lambda_k(c_k a_k)^p\Big)^{(t-p)/p}.\end{aligned} \tag{2.9}$$

Choosing $c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n} = (\Lambda_{n+1})^{\Lambda_n/t} \quad (n \in N)$ and setting $\Lambda_0 = 0$, from $\lambda_{n+1} \leq \lambda_n$, we have

$$\begin{aligned}c_n = \Big[\frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}\Big]^{1/t\lambda_n} &= \Big(1+\frac{\lambda_{n+1}}{\Lambda_n}\Big)^{\Lambda_n/t\lambda_n}\cdot\Lambda_n^{1/t}\\ &\leq \Big(1+\frac{\lambda_n}{\Lambda_n}\Big)^{\Lambda_n/t\lambda_n}\cdot\Lambda_n^{1/t}.\end{aligned}$$

This implies that

$$\begin{aligned}&\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{t/\Lambda_n}\\ &\leq \frac{t}{p}\sum_{m=1}^{\infty}\Big[\Big(1+\frac{1}{\Lambda_m/\lambda_m}\Big)^{\Lambda_m/\lambda_m}\Big]^{p/t}\lambda_m(a_m)^p\Lambda^{(p-t)/t}\Big(\sum_{k=1}^{m}\lambda_k(c_k a_k)^p\Big)^{(t-p)/p}.\end{aligned}$$

Hence, by the above inequality and Lemma 2.1, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{t/\Lambda_n} &< \frac{te^{p/t}}{p}\sum_{m=1}^{\infty}\Big(1-\frac{6\lambda_m}{12\Lambda_m+11\lambda_m}\Big)^{p/t}\\ &\times\lambda_m(a_m)^p\Lambda_m^{(p-t)/t}\Big(\sum_{k=1}^{m}\lambda_k(c_k a_k)^p\Big)^{(t-p)/p}.\end{aligned}$$

Thus the inequality (2.6) is proved. □

Remark. Setting $t \equiv 1$ in Theorem 2.7, then from (2.6), we obtain the inequality (2.2) in Theorem 2.2. Hence the inequality (2.6) is a new generalization of Hardy's inequality.

Moreover, we can consider a generalization version of the inequality (2.5) as following theorem:

Theorem 2.8. *Let* $0 < \lambda_{n+1} \le \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m (\Lambda_n \ge 1)$, $a_n \ge 0 (n \in N)$ *and* $0 < \sum_{n=1}^{\infty} \lambda_n (a_n)^t < \infty$ *for* $0 < p \le t < \infty$. *Then*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n} \\
&< \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \Big(1 - \frac{\lambda_n}{2(\Lambda_n + \lambda_n)} - \frac{\lambda_n^2}{24(\Lambda_n + \lambda_n)^2} - \frac{\lambda_n^3}{48(\Lambda_n + \lambda_n)^3}\Big)^{p/t} \\
&\quad \times \lambda_n (a_n)^p \Lambda_n^{(p-t)/t} \Big(\sum_{k=1}^{n} \lambda_k c_k a_k\Big)^{(t-p)/p}.
\end{aligned}$$

Proof. The proof is similar to the proof of theorem 2.7. □

Also assuming that $\lambda_n = 1$ in the Theorem 2.7 and Theorem 2.8, we have further extension of the strengthened Carleman's inequality as following:

Corollary 2.9. *Let* $a_n \ge 0 (n \in N)$, $0 < p \le 1$ *and* $0 < \sum_{n=1}^{\infty} a_n < \infty$. *Then*

$$\begin{aligned}
&\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{t/n} \\
&< \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \Big(1 - \frac{6}{12n + 11}\Big)^{p/t} (a_n)^p n^{(p-t)/t} \Big(\sum_{k=1}^{n} (c_k a_k)^p\Big)^{(t-p)/p}.
\end{aligned}$$

where $c_k = (1 + 1/k)^k \cdot k$.

Corollary 2.10. *Let* $a_n \ge 0 (n \in N)$, $0 < p \le 1$ *and* $0 < \sum_{n=1}^{\infty} a_n < \infty$. *Then*

$$\begin{aligned}
&\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{t/n} \\
&< \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \Big(1 - \frac{1}{2(1+n)} - \frac{1}{24(1+n)^2} - \frac{1}{48(1+n)^3}\Big)^{p/t} \\
&\quad \times (a_n)^p n^{(p-t)/t} \Big(\sum_{k=1}^{n} (c_k a_k)^p\Big)^{(t-p)/p}.
\end{aligned}$$

where $c_k = (1 + 1/k)^k \cdot k$.

References

[1] E. F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, Berlin/New York, 1961.
[2] G. S. Davis and G. M. Peterson, *On an inequality of Hardy's (II)*, Quart. J. Math. (Oxford) **15** (1964), 35-40.
[3] G. H. Hardy, *Note on a theorem of Hilbert*, Math. Zeitschr **6** (1920), 314-317.
[4] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge Univ. Press, London, 1952.
[5] Y. -H. Kim, *Refinements and Extensions of an inequality*, J. Math. Anal. Appl. **245** (2000), 628-632.
[6] J. Németh, *Generalizations of the Hardy-Littlewood inequality*, Acta. Sci. Math. (Szeged) **32** (1971), 295-299.
[7] Z. Xie and Y. Zhong, *A best approximation for constant e and an improvement to Hardy's inequality*, J. Math. Anal. Appl. **234** (1999), 717-722.
[8] B. C. Yang, *On Hardy's inequality*, J. Math. Anal. Appl. **234** (1999), 717-722.
[9] B. C. Yang and L. Debnath, *Some Inequalities Involving the Constant e, and an Application to Carleman's inequality*, J. Math. Anal. Appl. **223** (1998), 347-353.
[10] P. Yang and G. Z. Sun, *A Strengthened Carleman's inequality*, J. Math. Anal. Appl. **240** (1999), 290-193.
[11] X. Yang, *A Carleman's inequality*, J. Math. Anal. Appl. **253** (2001), 691-694.

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