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THE TRIANGLE INEQUALITY IN C^* ALGEBRAS

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Abstract. The triangle inequality fails comprehensively in C^* algebras, holding neither for commuting pairs of non normals nor for non commuting pairs of hermitians.

1. In both vector lattices and C^* algebras X there is a "modulus", or "absolute value", in the sense of a mapping $|\cdot| : X \to X$ with familiar properties: for arbitrary $x, y \in X$ and $\lambda \in \mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$

1.1 $$|\,|x| + |y|\,| = |x| + |y|$$

and

1.2 $$|\lambda x| = |\lambda|\,|x|\ .$$

If X is a real vector lattice [3], with partial order $\leq$ and least upper bound $\vee$, then

1.3 $$|x| = x \vee -x = x^+ + x^- \ \textit{where}\ x^+ = x \vee 0\ ,\ x^- = (-x)^+\ .$$

If X is a C^* algebra [2],[4], with multiplication $x, y \mapsto xy$ and involution $x \mapsto x^*$, then

1.4 $$|x| = (x^* x)^{1/2}\ .$$

In either case we shall write

1.5 $$x \leq y \iff y - x = |y - x| \iff y - x \in X^+\ .$$

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In particular $X = C(\Omega)$, the continuous functions on a compact Hausdorff space Ω, is both a vector lattice and a C^* algebra; here $|x|$ is given by

1.6 $$|x|(t) = |x(t)| \ (t \in \Omega) \ .$$

2. Of interest is the status of the *triangle inequality*,

2.1 $$|x + y| \leq |x| + |y| \ .$$

This is easily checked in real, and hence in complexified, vector lattices:

$$x \leq |x| \ , \ y \leq |y| \implies x + y \leq |x| + |y|$$

and

$$-x \leq |x| \ , \ -y \leq |y| \implies -(x + y) \leq |x| + |y| \ .$$

For continuous functions (1.6) the triangle inequality is very clear. It therefore [2] follows indirectly (Gelfand-Naimark) that the triangle inequality (2.1) holds in commutative C^* algebras, and more generally (Fuglede) for pairs of commuting normal elements $x, y \in X$. It is equally clear (1.1) that the triangle inequality holds for arbitrary pairs of positive elements. In general however failure of the triangle inequality in C^* algebras is comprehensive; (2.1) holds neither for commuting non normal, nor for non commuting hermitian, 2×2 matrices:

3. Example *If $x, y \in X = \mathbf{C}^{2\times 2}$ are given by either*

3.1 $$x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

or

3.2 $$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

then the triangle inequality (2.1) *fails.*

Proof. For hermitian matrices $x = x^* \in \mathbf{C}^{2\times 2}$

3.3 $$x \in X^+ \iff \min(\mathrm{tr}(x), \det(x)) \geq 0 \ .$$

We have in each of (3.1) and (3.2)

3.4 $$|x| + |y| = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \ , \ |x + y| = \frac{1}{\sqrt{5}} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \ ,$$

so that

$$|x| + |y| - |x + y| = \frac{1}{\sqrt{5}} \begin{pmatrix} 2\sqrt{5} - 3 & -1 \\ -1 & \sqrt{5} - 2 \end{pmatrix}$$

with positive trace, but determinant

$$\det(|x| + |y| - |x + y|) = \frac{1}{\sqrt{5}}(3\sqrt{5} - 7) = \frac{45 - 49}{\sqrt{5}(3\sqrt{5} + 7)} < 0 \ \bullet$$

The second part of (3.4) is easily confirmed: the right hand side is a positive matrix whose square is $w = (x+y)^*(x+y)$. To find it in the first place either diagonalise w, or search for positive matrices of the form $sw+t$ with square w. Bhatia ([1] Exercise V.1.11) gives an alternative to (3.2).

For mutually commuting normal $x, y \in X$ the analagous product inequality is clear:

3.5 $$|xy| \leq |x|\ |y|\ .$$

This will always fail for hermitian elements which do not commute (since their product cannot then be hermitian), and is liable to fail for commuting elements which are not normal. With for example $X = \mathbf{C}^{2\times 2}$ and

3.6 $$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ ,\ y = x + 1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\ ,\ xy = x$$

we find

3.7 $$|xy| = |x| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ ,\ |y| = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\ ,\ |x|\ |y| = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \neq |y|\ |x|\ .$$

Here we do not even get

3.8 $$|x|\ |y| + |y|\ |x| \geq |xy| + |yx|\ .$$

References

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