Faculty of Sciences and Mathematics
University of Niš

Available at:
www.pmf.ni.ac.yu/sajt/publikacije/publikacije_pocetna.html

Filomat **20:2** (2006), 67–80

ON A GENERALIZATION OF NORMAL, ALMOST NORMAL AND MILDLY NORMAL SPACES II

ERDAL EKICI[1] AND TAKASHI NOIRI

Abstract. The aim of this paper is to study further characterizations and the relationships of δp-normal spaces, almost δp-normal spaces and mildly δp-normal spaces. We introduce the notion of $g\delta pr$-closed sets. Also, we obtain properties of $g\delta pr$-closed sets and the relationships between $g\delta pr$-closed sets and the related generalized closed sets. By using $g\delta pr$-closed sets, we introduce new forms of generalized δ-precontinuity. Moreover, we obtain new characterizations of δp-normal spaces, almost δp-normal spaces and mildly δp-normal spaces and preservation theorems.

1. Introduction

In 2005, Ekici and Noiri [6] has introduced the notions of δp-normal spaces, almost δp-normal spaces and mildly δp-normal spaces. It is well known that separation axioms on topological spaces are important and basic subjects in studies of general topology and several branches of mathematics. In the literature, separation axioms have been researched by many mathematicians. Moreover, many authors have obtained many characterizations and generalizations of separation axioms by using generalized closed sets. In 1970, the first step of generalizing closed sets was done by Levine [9]. After that time, many authors have introduced and studied the relationships between separation axioms and generalized closed sets [13, 14]. The notions of generalized closed sets have been investigated extensively by many authors

[1]Corresponding author.
[2]Received: July 10, 2006

because the notion of generalized closed sets is a natural generalization of closed sets.

The purpose of this paper is to introduce a new class of generalized closed sets, namely $g\delta pr$-closed sets, which is a generalizing of $g\delta p$-closed sets and gpr-closed sets. The relations with other notions connected with $g\delta pr$-closed sets and also properties of $g\delta pr$-closed sets are investigated. Also, we introduce and study δp-regular $T_{1/2}$ spaces and new forms of generalized δ-precontinuous functions. As applications, using the notions of $g\delta pr$-closed sets, we obtain the further characterizations and properties of almost δp-normal spaces and mildly δp-normal spaces.

2. Preliminaries

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X . The closure of A and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively.

Definition 1. *A subset A of a space X is said to be:*
(1) regular open [19] if $A = int(cl(A))$,
(2) α-open [12] if $A \subset int(cl(int(A)))$,
(3) preopen [11] or nearly open [7] if $A \subset int(cl(A))$.

The complement of a preopen (resp. regular open) set is called preclosed [11] (resp. regular closed [19]). The intersection of all preclosed sets containing A is called the preclosure of A and is denoted by $pcl(A)$. The preinterior of A, denoted by $pint(A)$ is defined to be the union of all preopen sets contained in A.

The δ-interior [20] of a subset A of X is defined by the union of all regular open sets of X contained in A and is denoted by δ-$int(A)$. A subset A is called δ-open [20] if $A = \delta\text{-}int(A)$, i. e. a set is δ-open if it is the union of regular open sets. The complement of a δ-open set is called δ-closed. Alternatively, a set A of (X, τ) is called δ-closed [20] if $A = \delta\text{-}cl(A)$, where $\delta\text{-}cl(A) = \{x \in X : A \cap int(cl(U)) \neq \emptyset, U \in \tau \text{ and } x \in U\}$.

A subset A of a space X is said to be δ-preopen [17] if $A \subset int(\delta\text{-}cl(A))$. The complement of a δ-preopen set is said to be δ-preclosed. The intersection of all δ-preclosed sets of X containing A is called the δ-preclosure [17] of A and is denoted by δ-$pcl(A)$. The union of all δ-preopen sets of X contained in A is called δ-preinterior of A and is denoted by δ-$pint(A)$ [17]. A subset U of X is called a δ-preneighborhood of a point $x \in X$ if there exists a δ-preopen set V such that $x \in V \subset U$. Note that $\delta\text{-}pcl(A) = A \cup cl(\delta\text{-}int(A))$ and $\delta\text{-}pint(A) = A \cap int(\delta\text{-}cl(A))$.

The family of all δ-preopen (resp. δ-preclosed, α-open, δ-open, δ-closed) sets of a space X is denoted by $\delta PO(X)$ (resp. $\delta PC(X)$, $\alpha O(X)$, $\delta O(X)$, $\delta C(X)$).

Definition 2. *A function $f : X \to Y$ is called*

(1) almost continuous [18] (resp. R-map [4], completely continuous [1]) if $f^{-1}(V)$ is open (resp. regular open, regular open) in X for every regular open (resp. regular open, open) set V of Y,

(2) rc-preserving [13] (resp. almost closed [18]) if $f(F)$ is regular closed (resp. closed) in Y for every regular closed set F of X.

Definition 3. *A subset A of a space X is called generalized closed [9] (briefly, g-closed) if $cl(A) \subset U$ whenever $A \subset U$ and U is open in X.*

Definition 4. *A subset A of a space X is called generalized p-closed [10] (briefly, gp-closed) if $pcl(A) \subset U$ whenever $A \subset U$ and U is open in X.*

Definition 5. *A subset A of a space X is called regular generalized closed [16] (briefly, rg-closed) if $cl(A) \subset U$ whenever $A \subset U$ and U is regular open in X.*

Definition 6. *A subset A of a space X is called generalized preregular closed [8] or regular generalized preclosed [14] (briefly, gpr-closed) if $pcl(A) \subset U$ whenever $A \subset U$ and U is regular open in X.*

The complement of a *gp*-closed (resp. rg-closed, gpr-closed) set is called *gp*-open (resp. rg-open, gpr-open).

3. $g\delta pr$-closed sets

Definition 7. *A subset A of a space X is called generalized δ-preclosed (briefly, $g\delta p$-closed) [6] if δ-$pcl(A) \subset U$ whenever $A \subset U$ and U is open in X.*

Definition 8. *A subset A of a space X is called generalized δp-regular closed (briefly, $g\delta pr$-closed) if δ-$pcl(A) \subset U$ whenever $A \subset U$ and U is regular open in X.*

The complement of a $g\delta p$-closed (resp. $g\delta pr$-closed) set is called $g\delta p$-open (resp. $g\delta pr$-open).

The family of all generalized δp-regular open (resp. generalized δp-regular closed) sets of a space X is denoted by $G\delta PRO(X)$ (resp. $G\delta PRC(X)$).

Remark 9. *For a subset A of a topological space (X, τ), the following diagram holds:*

$$\begin{array}{ccccc}
\textit{closed} & \to & g\text{-}\textit{closed} & \to & rg\text{-}\textit{closed} \\
\downarrow & & \downarrow & & \downarrow \\
\textit{preclosed} & \to & gp\text{-}\textit{closed} & \to & gpr\text{-}\textit{closed} \\
\downarrow & & \downarrow & & \downarrow \\
\delta\text{-}\textit{preclosed} & \to & g\delta p\text{-}\textit{closed} & \to & g\delta pr\text{-}\textit{closed}
\end{array}$$

None of these implications is reversible as shown by the following examples.

Example 10. *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. *Then the sets* $\{a\}$ *and* $\{b\}$ *are* $g\delta pr$*-closed but not gpr-closed.*

Example 11. *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. *Then the sets* $\{a, c\}$ *and* $\{b, c\}$ *are* $g\delta pr$*-closed but not* $g\delta p$*-closed.*

For the other implications the examples can be seen in [6, 8-11, 14, 16, 17].

Definition 12. *A topological space* X *is said to be almost* δp*-regular if for each regular closed set* A *of* X *and each point* $x \in X \backslash A$, *there exist disjoint* δ*-preopen sets* U *and* V *such that* $x \in U$ *and* $A \subset V$.

Definition 13. *A subset* A *of a topological space* X *is said to be* δ*-preclosed relative to* X *if for every cover* $\{V_\alpha : \alpha \in \Lambda\}$ *of* A *by* δ*-preopen subsets of* X, *there exists a finite subset* Λ_0 *of* Λ *such that* $A \subset \cup\{\delta\text{-}pcl(V_\alpha) : \alpha \in \Lambda_0\}$.

Theorem 14. *If a space* X *is almost* δp*-regular and a subset* A *of* X *is* δ*-preclosed relative to* X, *then* A *is* $g\delta pr$*-closed.*

Proof. Let U be any regular open set of X containing A. For each $x \in A$, there exists a δ-preopen set $V(x)$ such that $x \in V(x) \subset \delta\text{-}pcl(V(x)) \subset U$. Since $\{V(x) : x \in A\}$ is a δ-preopen cover of A, there exists a finite subset A_0 of A such that $A \subset \cup\{\delta\text{-}pcl(V(x)) : \alpha \in A_0\}$. Hence we obtain $A \subset \delta - pcl(A) \subset \cup\{\delta\text{-}pcl(V(x)) : \alpha \in A_0\} \subset U$. This shows that A is $g\delta pr$-closed. □

Theorem 15. *A subset* A *of a space* X *is* $g\delta pr$*-open if and only if* $M \subset \delta$*-*$pint(A)$ *whenever* M *is regular closed and* $M \subset A$.

Proof. $(\Rightarrow)$: Let M be a regular closed set of X and $M \subset A$. Then $X \backslash M$ is regular open and $X \backslash A \subset X \backslash M$. Since $X \backslash A$ is $g\delta pr$-closed, $\delta\text{-}pcl(X \backslash A) \subset X \backslash M$, i.e. $X \backslash \delta\text{-}pint(A) \subset X \backslash M$. Hence, $M \subset \delta\text{-}pint(A)$.

$(\Leftarrow)$: Let G be a regular open set of X and $X \backslash A \subset G$. Since $X \backslash G$ is a regular closed set contained in A, by hypothesis $X \backslash G \subset \delta\text{-}pint(A)$, i.e. $X \backslash \delta\text{-}pint(A) = \delta\text{-}pcl(X \backslash A) \subset G$. Hence, $X \backslash A$ is $g\delta pr$-closed and so A is $g\delta pr$-open. □

Theorem 16. *A subset* A *of a space* X *is* $g\delta p$*-open if and only if* $F \subset \delta$*-*$pint(A)$ *whenever* F *is closed and* $F \subset A$.

Lemma 17. *Let* A *and* X_0 *be subsets of a space* (X, τ). *If* $A \in \delta PO(X)$ *and* $X_0 \in \delta O(X)$, *then* $A \cap X_0 \in \delta PO(X_0)$ *[17].*

Lemma 18. *Let* $A \subset X_0 \subset X$. *If* $X_0 \in \delta O(X)$ *and* $A \in \delta PO(X_0)$, *then* $A \in \delta PO(X)$ *[17].*

Theorem 19. *If Y is a regular open subspace of a space X and A is a subset of Y, then δ-$pcl_Y(A) = \delta$-$pcl(A) \cap Y$.*

Theorem 20. *If A is a regular open and $g\delta pr$-closed subset of a space X, then A is δ-preclosed in X.*

Proof. If A is regular open and $g\delta pr$-closed, then δ-$pcl(A) \subset A$. This implies A is δ-preclosed. □

Theorem 21. *Let Y be a regular open subspace of a space X and $A \subset Y$. If A is $g\delta pr$-closed in X, then A is $g\delta pr$-closed in Y.*

Proof. Let U be a regular open set of Y such that $A \subset U$. Then $U = V \cap Y$ for some regular open set V of X. Since A is $g\delta pr$-closed in X, we have δ-$pcl(A) \subset V$ and by Theorem 19, δ-$pcl_Y(A) = \delta$-$pcl(A) \cap Y \subset V \cap Y = U$. Hence A is $g\delta pr$-closed in Y. □

Theorem 22. *Let Y be a $g\delta pr$-closed and regular open subspace of a space X. If A is $g\delta pr$-closed in Y, then A is $g\delta pr$-closed in X.*

Proof. Let U be any regular open subset of X such that $A \subset U$. Since $U \cap Y$ is regular open in Y and A is $g\delta pr$-closed in Y, δ-$pcl_Y(A) \subset U \cap Y$. By Theorem 19 and 20, we have δ-$pcl(A) = \delta$-$pcl(A) \cap Y = \delta$-$pcl_Y(A) \subset U \cap Y \subset U$. Hence, A is $g\delta pr$-closed in X. □

Theorem 23. *Let Y be a regular open and $g\delta pr$-closed subspace of a space X. If A is $g\delta pr$-open in Y, then A is $g\delta pr$-open in X.*

Proof. Let M be any regular closed set and $M \subset A$. Since M is regular closed in Y and A is $g\delta pr$-open in Y, $M \subset \delta$-$pint_Y(A)$ and then $M \subset \delta$-$pint(A) \cap Y$. Hence $M \subset \delta$-$pint(A)$ and so A is $g\delta pr$-open in X. □

Theorem 24. *If a subset A of a topological space X is $g\delta pr$-closed, then δ-$pcl(A) \backslash A$ contains no nonempty regular closed set in X.*

Proof. Suppose that there exists a nonempty regular closed set M of X such that $M \subset \delta$-$pcl(A) \backslash A$. Since $X \backslash M$ is regular open and A is $g\delta pr$-closed, δ-$pcl(A) \subset X \backslash M$, i.e. $M \subset X \backslash \delta$-$pcl(A)$. Then $M \subset \delta - pcl(A) \cap (X \backslash \delta$-$pcl(A)) = \varnothing$ and hence $M = \varnothing$, which is a contradiction. □

Theorem 25. *A $g\delta pr$-closed subset A of a topological space X is δ-preclosed if and only if δ-$pcl(A) \backslash A$ is regular closed.*

Proof. Let A be a $g\delta pr$-closed subset of X. Since δ-$pcl(A) \backslash A$ is regular closed, by the previous theorem, δ-$pcl(A) \backslash A = \varnothing$. Hence, A is δ-preclosed.

Conversely, if $g\delta pr$-closed set A is δ-preclosed, then δ-$pcl(A) \backslash A = \varnothing$ and hence δ-$pcl(A) \backslash A$ is a regular closed. □

Theorem 26. *If a subset A of a topological space X is $g\delta pr$-open, then $G = X$, whenever G is regular open and δ-$pint(A) \cup (X \backslash A) \subset G$.*

Proof. Let G be a regular open set of X and δ-$pint(A) \cup (X \backslash A) \subset G$. Then $X \backslash G \subset (X \backslash \delta\text{-}pint(A)) \cap A = \delta\text{-}pcl(X \backslash A) \backslash (X \backslash A)$. Since $X \backslash G$ is regular closed and $X \backslash A$ is $g\delta pr$-closed, by Theorem 24, $X \backslash G = \varnothing$ and hence $G = X$. □

Theorem 27. *If a subset A is $g\delta pr$-closed in a topological space X, then δ-$pcl(A) \backslash A$ is $g\delta pr$-open.*

Proof. Let F be a regular closed set of X such that $F \subset \delta\text{-}pcl(A) \backslash A$. Since A is $g\delta pr$-closed, by Theorem 24, $F = \varnothing$ and hence $F \subset \delta\text{-}pint(\delta\text{-}pcl(A) \backslash A)$. By Theorem 15, δ-$pcl(A) \backslash A$ is $g\delta pr$-open. □

4. $g\delta pr$-closure of a set

Theorem 28. *Let A and B be $g\delta pr$-closed sets in (X, τ) such that $cl(A) = \delta$-$pcl(A)$ and $cl(B) = \delta$-$pcl(B)$. Then $A \cup B$ is $g\delta pr$-closed.*

Proof. Let $A \cup B \subset U$ where U is regular open. Then $A \subset U$ and $B \subset U$. Since A and B are $g\delta pr$-closed δ-$pcl(A) \subset U$ and δ-$pcl(B) \subset U$. Now, $cl(A \cup B) = cl(A) \cup cl(B) = \delta\text{-}pcl(A) \cup \delta\text{-}pcl(B) \subset U$. But $\delta\text{-}pcl(A \cup B) \subset cl(A \cup B)$. So $\delta\text{-}pcl(A \cup B) \subset U$ and hence $A \cup B$ is $g\delta pr$-closed. □

Theorem 29. *If A is $g\delta pr$-closed and $A \subset B \subset \delta$-$pcl(A)$, then B is $g\delta pr$-closed.*

Proof. Let $B \subset U$ where U is regular open. Then $A \subset B$ implies $A \subset U$. Since A is $g\delta pr$-closed, δ-$pcl(A) \subset U$. $B \subset \delta\text{-}pcl(A)$ implies $\delta\text{-}pcl(B) \subset \delta$-$pcl(A)$. Thus, δ-$pcl(B) \subset U$ and this shows that B is $g\delta pr$-closed. □

Theorem 30. *If δ-$pint(A) \subset B \subset A$ and A is $g\delta pr$-open, then B is $g\delta pr$-open.*

Proof. δ-$pint(A) \subset B \subset A$ implies $X \backslash A \subset X \backslash B \subset X \backslash \delta\text{-}pint(A)$. That is, $X \backslash A \subset X \backslash B \subset \delta - pcl(X \backslash A)$. Since $X \backslash A$ is $g\delta pr$-closed, by Theorem 29, $X \backslash B$ is $g\delta pr$-closed and B is $g\delta pr$-open. □

Definition 31. *Let (X, τ) be a topological space and $A \subset X$. The almost kernel of A, denoted by a-$ker(A)$, is the intersection of all regular open supersets of A.*

Theorem 32. *A subset A of a topological space X is $g\delta pr$-closed if and only if δ-$pcl(A) \subset a$-$ker(A)$.*

Proof. Since A is $g\delta pr$-closed, δ-$pcl(A) \subset G$ for any regular open set G with $A \subset G$ and hence δ-$pcl(A) \subset a$-$ker(A)$.

Conversely, let G be any regular open set such that $A \subset G$. By hypothesis, $\delta\text{-}pcl(A) \subset a\text{-}ker(A) \subset G$ and hence A is $g\delta pr$-closed. □

Definition 33. *For a subset A of a topological space (X, τ), $g\delta pr$-$cl(A) = \cap\{F : A \subset F,\ F$ is $g\delta pr$-closed in $X\}$.*

Theorem 34. *For a $x \in X$, $x \in g\delta pr$-$cl(A)$ if and only if $V \cap A \neq \varnothing$ for every $g\delta pr$-open set V containing x.*

Proof. Suppose that there exists a $g\delta pr$-open set V containing x such that $V \cap A = \varnothing$. Since $A \subset X\backslash V$, $g\delta pr$-$cl(A) \subset X\backslash V$. This implies $x \notin g\delta pr$-$cl(A)$, a contradiction.

Conversely, suppose that $x \notin g\delta pr$-$cl(A)$. Then there exists a $g\delta pr$-closed subset F containing A such that $x \notin F$. Then $x \in X\backslash F$ and $X\backslash F$ is $g\delta pr$-open. Also $(X\backslash F) \cap A = \varnothing$, a contradiction. □

Theorem 35. *Let A and B be subsets of a topological space (X, τ). Then:*
(1) $g\delta pr$-$cl(\varnothing) = \varnothing$ and $g\delta pr$-$cl(X) = X$,
(2) If $A \subset B$, then $g\delta pr$-$cl(A) \subset g\delta pr$-$cl(B)$,
(3) $A \subset g\delta pr$-$cl(A)$,
(4) $g\delta pr$-$cl(A) = g\delta pr$-$cl(g\delta pr - cl(A))$,
(5) $g\delta pr$-$cl(A \cup B) \supset g\delta pr$-$cl(A) \cup g\delta pr$-$cl(B)$,
(6) $g\delta pr$-$cl(A \cap B) \subset g\delta pr$-$cl(A) \cap g\delta pr$-$cl(B)$.

Remark 36. *If a subset A of a space X is $g\delta pr$-closed, then $g\delta pr$-$cl(A) = A$.*

Theorem 37. *Let (X, τ) be a topological space. If $G\delta PRO(X)$ is a topology, then $\tau^* = \{V \subset X : g\delta pr - cl(X\backslash V) = X\backslash V\} = G\delta PRO(X)$.*

Theorem 38. *For a topological space (X, τ), every $g\delta pr$-closed set is closed if and only if $\tau^* = \tau$.*

Proof. Let $A \in \tau^*$. Then $g\delta pr$-$cl(X\backslash A) = X\backslash A$. Since every $g\delta pr$-closed set is closed, $cl(X\backslash A) = g\delta pr$-$cl(X\backslash A) = X\backslash A$. Hence, $A \in \tau$.

Conversely, suppose $\tau^* = \tau$. Let A be a $g\delta pr$-closed set. Then $g\delta pr$-$cl(A) = A$. This implies $X\backslash A \in \tau^* = \tau$. So A is closed. □

5. δp-REGULAR $T_{1/2}$ SPACES AND GENERALIZED δ-PRECONTINUOUS FUNCTIONS

Definition 39. *A topological space X is called δp-regular $T_{1/2}$ if every $g\delta pr$-closed set is δ-preclosed.*

Theorem 40. *The following conditions are equivalent for a topological space X:*
(a) X is δp-regular $T_{1/2}$,
(b) Every singleton is either regular closed or δ-preopen.

Proof. $(a) \Rightarrow (b)$: Let $x \in X$ and assume that $\{x\}$ is not regular closed. Then $X\backslash\{x\}$ is not regular open and hence $X\backslash\{x\}$ is trivially $g\delta pr$-closed. By (a), it is δ-preclosed and hence $\{x\}$ is δ-preopen.

(b) $\Rightarrow$ *(a)* : Let $A \subset X$ be $g\delta pr$-closed. Let $x \in \delta$-$pcl(A)$. We will show that $x \in A$. For, consider the following two cases:

Case (i)- The set $\{x\}$ is regular closed. Then, if $x \notin A$, there exists a regular closed set in δ-$pcl(A) \backslash A$. By Theorem 24, $x \in A$.

Case (ii)- The set $\{x\}$ is δ-preopen. Since $x \in \delta$-$pcl(A)$, then $\{x\} \cap A \neq \varnothing$. Thus, $x \in A$.

So, in both cases, $x \in A$. This shows that δ-$pcl(A) \subset A$ or equivalently A is δ-preclosed. □

Theorem 41. *For a topological space (X, τ), the following hold:*

(1) $\delta PO(X) \subset G\delta PRO(X)$,

(2) The space X is δp-regular $T_{1/2}$ if and only if $\delta PO(X) = G\delta PRO(X)$.

Proof. (1) Let A be δ-preopen. Then $X \backslash A$ is δ-preclosed and so $g\delta pr$-closed. This implies that A is $g\delta pr$-open. Hence, $\delta PO(X) \subset G\delta PRO(X)$.

(2) ($\Rightarrow$) : Let X be δp-regular $T_{1/2}$. Let $A \in G\delta PRO(X)$. Then $X \backslash A$ is $g\delta pr$-closed. By hypothesis, $X \backslash A$ is δ-preclosed and thus $A \in \delta PO(X)$. Hence, $G\delta PRO(X) = \delta PO(X)$.

($\Leftarrow$) : Let $\delta PO(X) = G\delta PRO(X)$. Let A be $g\delta pr$-closed. Then $X \backslash A$ is $g\delta pr$-open. Hence, $X \backslash A \in \delta PO(X)$. Thus, A is δ-preclosed thereby implying X is δp-regular $T_{1/2}$. □

Definition 42. *A function $f : X \to Y$ is called:*

(1) $g\delta p$-continuous if $f^{-1}(F)$ is $g\delta p$-closed in X for every closed set F of Y,

(2) δp-$g\delta p$-continuous [6] if $f^{-1}(F)$ is $g\delta p$-closed in X for every δ-preclosed set F of Y,

(3) $g\delta p$-irresolute if $f^{-1}(F)$ is $g\delta p$-closed in X for every $g\delta p$-closed set F of Y.

Definition 43. *A function $f : X \to Y$ is called:*

(1) $g\delta pr$-continuous if $f^{-1}(F)$ is $g\delta pr$-closed in X for every closed set F of Y,

(2) δp-$g\delta pr$-continuous if $f^{-1}(F)$ is $g\delta pr$-closed in X for every δ-preclosed set F of Y,

(3) $g\delta pr$-irresolute if $f^{-1}(F)$ is $g\delta pr$-closed in X for every $g\delta pr$-closed set F of Y.

Definition 44. *A function $f : X \to Y$ is called:*

(1) δ-precontinuous [17] if $f^{-1}(F)$ is δ-preclosed in X for every closed set F of Y,

(2) δ-preirresolute [5] if $f^{-1}(F)$ is δ-preclosed in X for every δ-preclosed set F of Y.

Remark 45. *The following diagram holds for a function $f : (X, \tau) \to (Y, \sigma)$:*

$$\begin{array}{ccccc}
g\delta pr\text{-irresoluteness} & \Rightarrow & \delta p\text{-}g\delta pr\text{-continuity} & \Rightarrow & g\delta pr\text{-continuity} \\
 & & \Uparrow & & \Uparrow \\
g\delta p\text{-irresoluteness} & \Rightarrow & \delta p\text{-}g\delta p\text{-continuity} & \Rightarrow & g\delta p\text{-continuity} \\
 & & \Uparrow & & \Uparrow \\
 & & \delta\text{-preirresoluteness} & \Rightarrow & \delta\text{-precontinuity}
\end{array}$$

None of these implications is reversible as shown by the following examples.

Example 46. *Let $X = Y = \{a, b, c, d\}$ and $\tau = \sigma = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined as follows: $f(a) = a$, $f(b) = b$, $f(c) = b$ and $f(d) = d$. Then f is $g\delta pr$-continuous but it is not $g\delta p$-continuous.*

If we define the function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = b$, $f(b) = d$, $f(c) = a$ and $f(d) = d$, then f is $g\delta pr$-continuous but it is not δp-$g\delta pr$-continuous. Also, f is δ-precontinuous but it is not δ-preirresolute.

Example 47. *Let $X = Y = \{a, b, c, d\}$ and $\tau = \sigma = \{X, \varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined as follows: $f(a) = d$, $f(b) = a$, $f(c) = b$ and $f(d) = d$. Then f is δp-$g\delta pr$-continuous but it is not $g\delta pr$-irresolute.*

If we define the function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = c$, $f(b) = c$, $f(c) = a$ and $f(d) = a$, then f is δp-$g\delta pr$-continuous but it is not δp-$g\delta p$-continuous.

Example 48. *Let $X = Y = \{a, b, c, d\}$ and $\tau = \sigma = \{X, \varnothing, \{b\}, \{d\}, \{b, d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined as follows: $f(a) = a$, $f(b) = c$, $f(c) = b$ and $f(d) = d$. Then f is $g\delta p$-continuous but it is neither δp-$g\delta p$-continuous nor δ-precontinuous.*

If we define the function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = a$, $f(b) = b$, $f(c) = d$ and $f(d) = b$, then f is δp-$g\delta p$-continuous but it is neither $g\delta p$-irresolute nor δ-preirresolute.

Theorem 49. *Let $f : X \to Y$ and $g : Y \to Z$ be functions.*

(1) If f is $g\delta pr$-irresolute and g is $g\delta pr$-continuous, then the composition $gof : X \to Z$ is $g\delta pr$-continuous.

(2) If f is $g\delta pr$-irresolute and g is δp-$g\delta pr$-continuous, then the composition $gof : X \to Z$ is δp-$g\delta pr$-continuous.

(3) If f and g are δp-$g\delta pr$-continuous and Y is δp-regular $T_{1/2}$, then the composition $gof : X \to Z$ is δp-$g\delta pr$-continuous.

(4) If f and g are $g\delta pr$-irresolute, then the composition $gof : X \to Z$ is $g\delta pr$-irresolute.

Theorem 50. *If a function $f : X \to Y$ is δp-$g\delta pr$-continuous and Y is δp-regular $T_{1/2}$, then f is $g\delta pr$-irresolute.*

Proof. Let F be any $g\delta pr$-closed subset of Y. Since Y is δp-regular $T_{1/2}$, then F is δ-preclosed in Y. Hence $f^{-1}(F)$ is $rg\delta p$-closed in X. This show that f is $g\delta pr$-irresolute. □

Definition 51. *A topological space X is said to be*

(1) extremally disconnected [3] if the closure of each open set of X is open in X,

(2) submaximal [3] if every dense subset of X is open.

Definition 52. *Let (X, τ) be a topological space. The collection of all regular open sets forms a base for a topology τ^*. It is called the semiregularization. In case $\tau = \tau^*$, the space (X, τ) is called semi-regular [19].*

Theorem 53. *If a function $f : X \to Y$ is $g\delta pr$-continuous and Y is submaximal extremally disconnected and semi-regular, then f is δp-$g\delta pr$-continuous.*

Proof. Let F be any δ-preclosed subset of Y. Since Y is submaximal extremally disconnected semi-regular, then F is closed in Y. Hence $f^{-1}(F)$ is $rg\delta p$-closed in X. This show that f is δp-$g\delta pr$-continuous. □

Theorem 54. *If a function $f : X \to Y$ is $g\delta pr$-continuous and X is δp-regular $T_{1/2}$, then f is δ-precontinuous.*

Proof. Let F be any closed set of Y. Since f is $g\delta pr$-continuous, $f^{-1}(F)$ is $g\delta pr$-closed in X and then $f^{-1}(F)$ is δ-preclosed in X. Hence f is δ-precontinuous. □

Theorem 55. *If a function $f : X \to Y$ is δp-$g\delta pr$-continuous and X is δp-regular $T_{1/2}$, then f is δ-preirresolute.*

Proof. Let F be any δ-preclosed set of Y. Since f is δp-$g\delta pr$-continuous, $f^{-1}(F)$ is $g\delta pr$-closed in X and then $f^{-1}(F)$ is δ-preclosed in X. Hence f is δ-preirresolute. □

6. Almost δp-normal, mildly δp-normal spaces and preservation properties

Definition 56. *A function $f : X \to Y$ is called strongly δ-preclosed [6] if $f(U) \in \delta PC(Y)$ for each $U \in \delta PC(X)$.*

Definition 57. *A function $f : X \to Y$ is called*

(1) δp-$g\delta p$-closed [6] if $f(F)$ is $g\delta p$-closed in Y for every δ-preclosed set F of X,

(2) δp-$g\delta pr$-closed if $f(F)$ is $g\delta pr$-closed in Y for every δ-preclosed set F of X.

Definition 58. *A space X is said to be*

(1) δp-normal [6] if for every pair of disjoint closed sets A and B of X, there exist disjoint δ-preopen sets U and V such that $A \subset U$ and $B \subset V$,

(2) almost δp-normal [6] if for each closed set A and regular closed set B of X such that $A \cap B = \varnothing$, there exist disjoint δ-preopen sets U and V such that $A \subset U$ and $B \subset V$,

(3) mildly δp-normal [6] if for every pair of disjoint regular closed sets A and B of X, there exist disjoint δ-preopen sets U and V such that $A \subset U$ and $B \subset V$.

Theorem 59. *The following are equivalent for a space X:*

(1) X is almost δp-normal,

(2) For each closed set A and regular closed set B such that $A \cap B = \varnothing$, there exist disjoint $g\delta p$-open sets U and V such that $A \subset U$ and $B \subset V$,

(3) For each closed set A and each regular open set B containing A, there exists a $g\delta p$-open set V of X such that $A \subset V \subset \delta\text{-}pcl(V) \subset B$,

(4) For each rg-closed set A and each regular open set B containing A, there exists a δ-preopen set V of X such that $cl(A) \subset V \subset \delta\text{-}pcl(V) \subset B$,

(5) For each rg-closed set A and each regular open set B containing A, there exists a $g\delta p$-open set V of X such that $cl(A) \subset V \subset \delta\text{-}pcl(V) \subset B$,

(6) For each g-closed set A and each regular open set B containing A, there exists a δ-preopen set V of X such that $cl(A) \subset V \subset \delta\text{-}pcl(V) \subset B$,

(7) For each g-closed set A and each regular open set B containing A, there exists a $g\delta p$-open set V of X such that $cl(A) \subset V \subset \delta\text{-}pcl(V) \subset B$.

Proof. (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (5), (5) $\Rightarrow$ (4), (4) $\Rightarrow$ (6) and (6) $\Rightarrow$ (7) are obvious.

(2) $\Rightarrow$ (3) : Let A be a closed set and B be a regular open subset of X containing A. There exist disjoint $g\delta p$-open sets V and W such that $A \subset V$ and $X \backslash B \subset W$. By Theorem 16, we have $X \backslash B \subset \delta\text{-}pint(W)$ and $V \cap \delta\text{-}pint(W) = \varnothing$. Hence, we obtain $\delta\text{-}pcl(V) \cap \delta\text{-}pint(W) = \varnothing$ and hence $A \subset V \subset \delta\text{-}pcl(V) \subset X \backslash \delta\text{-}pint(W) \subset B$.

(7) $\Rightarrow$ (1) : Let A be any closed set and B be any regular closed set such that $A \cap B = \varnothing$. Then $X \backslash B$ is a regular open set containing A and there exists a $g\delta p$-open set G of X such that $A \subset G \subset \delta\text{-}pcl(G) \subset X \backslash B$. Put $U = \delta\text{-}pint(G)$ and $V = X \backslash \delta\text{-}pcl(G)$. Then U and V are disjoint δ-preopen sets of X such that $A \subset U$ and $B \subset V$. Hence X is almost δp-normal. □

Theorem 60. *The following are equivalent for a space X:*

(1) X is mildly δp-normal,

(2) For any disjoint regular closed sets A and B of X, there exist disjoint $g\delta p$-open sets U and V such that $A \subset U$ and $B \subset V$,

(3) For any disjoint regular closed sets A and B of X, there exist disjoint $g\delta pr$-open sets U and V such that $A \subset U$ and $B \subset V$,

(4) For each regular closed set A and each regular open set B containing A, there exists a $g\delta p$-open set V of X such that $A \subset V \subset \delta\text{-}pcl(V) \subset B$,

(5) For each regular closed set A and each regular open set B containing A, there exists a $g\delta pr$-open set V of X such that $A \subset V \subset \delta$-$pcl(V) \subset B$.

Proof. The proof is similar to Theorem 59 by using Theorem 29 in [6]. □

Theorem 61. *Let $f : (X, \tau) \to (Y, \sigma)$ be R-map and strongly δ-preclosed. Then for every $g\delta pr$-closed set A of X, $f(A)$ is $g\delta pr$-closed in Y.*

Proof. Let A be $g\delta pr$-closed in X. Let $f(A) \subset U$, where U is regular open in Y. Then $A \subset f^{-1}(U)$. Since f is R-map and A is $g\delta pr$-closed, δ-$pcl(A) \subset f^{-1}(U)$. That is, $f(\delta\text{-}pcl(A)) \subset U$. Now $\delta\text{-}pcl(f(A)) \subset \delta - pcl(f(\delta\text{-}pcl(A))) = f(\delta - pcl(A)) \subset U$, since f is strongly δ-preclosed. Hence, $f(A)$ is $g\delta pr$-closed in Y. □

Lemma 62. *A surjection $f : X \to Y$ is δp-$g\delta p$-closed (resp. δp-$g\delta pr$-closed) if and only if for each subset B of Y and each δ-preopen set U of X containing $f^{-1}(B)$ there exists a $g\delta p$-open (resp. $g\delta pr$-open) set of V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.*

Theorem 63. *If $f : X \to Y$ is a δp-$g\delta p$-closed continuous surjection and X is δp-normal, then Y is δp-normal.*

Proof. Let A and B be any disjoint closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of X. Since X is δp-normal, there exist disjoint δ-preopen sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. By Lemma 62, there exist $g\delta p$-open sets G and H of Y such that $A \subset G$, $B \subset H$, $f^{-1}(G) \subset U$ and $f^{-1}(H) \subset V$. Since U and V are disjoint, G and H are disjoint. By Theorem 16, we have $A \subset \delta\text{-}pint(G)$, $B \subset \delta\text{-}pint(H)$ and $\delta\text{-}pint(G) \cap \delta\text{-}pint(H) = \varnothing$. This shows that Y is δp-normal. □

The proofs of the following theorems are similar to previous one.

Theorem 64. *If $f : X \to Y$ is an R-map δp-$rg\delta p$-closed surjection and X is mildly δp-normal, then Y is mildly δp-normal.*

Theorem 65. *If $f : X \to Y$ is a completely continuous δp-$rg\delta p$-closed surjection and X is mildly δp-normal, then Y is δp-normal.*

Theorem 66. *If $f : X \to Y$ is an almost continuous δp-$rg\delta p$-closed surjection and X is δp-normal, then Y is mildly δp-normal.*

Theorem 67. *If f is a δp-$g\delta pr$-continuous rc-preserving injection and Y is mildly δp-normal, then X is mildly δp-normal.*

Proof. Let A and B be any disjoint regular closed sets of X. Since f is an rc-preserving injection, $f(A)$ and $f(B)$ are disjoint regular closed sets of Y. By mild δp-normality of Y, there exist disjoint δ-preopen sets U and V of Y such that $f(A) \subset U$ and $f(B) \subset V$. Since f is δp-$g\delta pr$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint $g\delta pr$-open sets containing A and B, respectively. Hence by Theorem 60, X is mildly δp-normal. □

Theorem 68. *If $f : X \to Y$ is a δp-$g\delta pr$-continuous almost closed injection and Y is δp-normal, then X is mildly δp-normal.*

Proof. Similar to previous one. □

References

[1] S. P. Arya and R. Gupta, On strongly continuous functions, Kyungpook Math. J., 14 (1974), 131-141.
[2] S. P. Arya and T. Nour, Characterizations of s-normal spaces, Indian J. Pure Appl. Math., 21 (1990), 717-719.
[3] N. Bourbaki, General Topology, Part I, Addison Wesley, Reading, Mass. 1996.
[4] D. Carnahan, Some properties related to compactness in topological spaces, Ph. D. Thesis, University of Arkansas, 1973.
[5] E. Ekici, (δ-pre,s)-continuous functions, Bull. Malays. Math. Sci. Soc. (2) (27) (2004), 237-251.
[6] E. Ekici and T. Noiri, On a generalization of normal, almost normal and mildly normal spaces-I, submitted.
[7] M. Ganster and I. L. Reilly, Locally closed sets and LC-continuous functions, Int. J. Math. Math. Sci., 3 (1989), 417-424.
[8] Y. Gnanambal, On generalized preregular closed sets in topological spaces, Indian J. Pure Appl. Math., 28 (1997), 351-360.
[9] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo (2), 19 (1970), 89-96.
[10] H. Maki, J. Umehara and T. Noiri, Every topological space is pre-$T_{1/2}$, Mem. Fac. Sci. Kochi Univ. Ser. A Math., 17 (1996), 33-42.
[11] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt, 53 (1982), 47-53.
[12] O. Njastad, On some classes of nearly open sets, Pacific J. Math., 15 (1965), 961-970.
[13] T. Noiri, Mildly normal spaces and some functions, Kyungpook Math. J., 36 (1996), 183-190.
[14] T. Noiri, Almost p-regular spaces and some functions, Acta Math. Hungar., 79 (1998), 207–216.
[15] T. Noiri, Almost αg-closed functions and separation axioms, Acta Math. Hungar., 82 (3) (1999), 193-205.
[16] N. Palaniappan and K. C. Rao, Regular generalized closed sets, Kyungpook Math. J., 33 (1993), 211-219.
[17] S. Raychaudhuri and N. Mukherjee, On δ-almost continuity and δ-preopen sets, Bull. Inst. Math. Acad. Sinica, 21 (1993), 357-366.
[18] M. K. Singal and A. R. Singal, Almost continuous mappings, Yokohama Math. J., 16 (1968), 63–73.
[19] M. H. Stone, Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., 41 (1937), 375-381.
[20] N. V. Velicko, H-closed topological spaces, Amer. Math. Soc. Transl., 78 (1968), 103-118.

Erdal Ekici:
Department of Mathematics, Canakkale Onsekiz Mart University
Terzioglu Campus, 17020 Canakkale, TURKEY
E-mail: eekici@comu.edu.tr

Takashi Noiri:
Department of Mathematics, Yatsushiro College of Techology
Yatsushiro, Kumamoto, 866-8501, JAPAN
E-mail: noiri@as.yatsushiro-nct.ac.jp