Faculty of Sciences and Mathematics
University of Niš

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GENERALIZATION OF TWO ASYMPTOTICALLY STATISTICAL EQUIVALENT THEOREMS

EKREM SAVAŞ AND RICHARD F. PATTERSON

Abstract. The goal of this paper is to present two theorems that characterize asymptotically statistical equivalent of multiple L and the regularity of asymptotically statistical convergence by using a sequence of infinite matrices.

1. Introduction and Background

In 1998 Kolk presented the notion of B-statistical convergence by considering a sequence of infinite matrices. In addition, the definition of asymptotically statistical equivalent sequences was presented in [6]. By combining the notions of B-statistical convergence and asymptotically statistical equivalent sequences we shall present answers to the following questions: Which type of summability matrices preserve asymptotically statistical equivalent of multiple L for a given sequence? What are the necessary and sufficient conditions that will ensure the regularity of asymptotically statistical for a given sequence? Let $l^1 = \{x = (x_k) : \sum_{k=1}^{\infty} |x_k| < \infty.\}$ and $d_A = \{x = (x_k) : \lim_n \sum_{k=1}^{\infty} a_{n,k}x_k = \text{ exists}\}$.

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Definition 1.1 (Fridy, [1]). *For each $x = (x_k)$ in l^1 the " remainder sequence" $[Rx]$ is the sequence whose n-th term is*

$$R_n x := \sum_{k \geq n} |x_k| \,.$$

Definition 1.2 (Marouf, [4]). *Two nonnegative sequences $x = (x_k)$, and $y = (y_k)$ are said to be asymptotically equivalent if*

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$).

Definition 1.3 (Fridy, [2]). *The sequence $x = (x_k)$ has statistical limit L provided that for every $\epsilon > 0$,*

$$\lim_n \frac{1}{n}\{\textit{the number of } k \leq n : |x_k - L| \geq \epsilon\} = 0.$$

The next definition is natural combination of definition (1.2) and (1.3).

Definition 1.4. *Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L, briefly $x \overset{S_L}{\sim} y$, if for every $\epsilon > 0$,*

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \epsilon \right\} \right| = 0,$$

where $|K|$ denotes the cardinality of K.

Let $P_\delta = \{x = (x_k) : x_k \geq \delta > 0$ for all $k\}$ and let P_0 be the set of all nonnegative sequences which have at most a finite number of zero entries.

For $x = (x_k) \in l^1$ let $Rx = (R_n x) = (\sum_{k \geq n} |x_k|)$. For a sequence x and an infinite matrix $A = (a_{n,k})$ let $Ax = (\sum_k a_{n,k} x_k)$ provided that all series $\sum_k a_{n,k} x_k$ converge.

Definition 1.5. *If $\boldsymbol{B} = (\boldsymbol{B}_i)$ is a sequence of infinite matrices $\boldsymbol{B}_i = (b_{n,k}(i))$, then a sequence $x \in l_\infty$ is said to be $\boldsymbol{B}$-summable to the value x_0 if $\lim_n (\boldsymbol{B}_i x)_n = \lim_n \sum_k b_{n,k}(i) x_k = x_0$, uniformly in i.*

Taking in the theorems of Patterson [6] a summability method **B** instead of summability matrix A the authors give two analogous theorems.

Definition 1.6. *A summability matrix $\boldsymbol{B} = (\boldsymbol{B}_i)$ is asymptotically statistical regular provided that $\boldsymbol{B}_i x \overset{S_L}{\sim} \boldsymbol{B}_i y$ whenever $x \overset{S_L}{\sim} y$, $x \in P_0$, and $y \in P_\delta$ for some $\delta > 0$.*

2. Main Results

In this section we shall present two theorems concerning necessary and sufficient conditions of the matrix transformation that will preserve asymptotically statistical equivalents of multiple L of a given sequence and the regularity of asymptotically statistical convergence.

Theorem 2.1. *If $\boldsymbol{B} = (\boldsymbol{B}_i)$ is a sequence of infinite nonnegative matrices with $\boldsymbol{B}_i = (b_{n,k}(i))$ that maps bounded sequence $x = (x_k)$ into l^1 then the following statements are equivalent:*

(1) *If $x = (x_k)$ and $y = (y_k)$ are sequences such that $x \overset{S^L}{\sim} y$, $x \in P_0$ and $y \in P_\delta$ for some $\delta > 0$ then*

$$R_n(\boldsymbol{B}_i x) \overset{S_L}{\sim} R_n(\boldsymbol{B}_i y).$$

(2)

$$\lim_n \frac{1}{n} \max_i \left| \left\{ k \leq n : \left| \frac{\sum_{p=k}^{\infty} b_{p,m}(i)}{\sum_{p=k}^{\infty} \sum_{j=1}^{\infty} b_{p,j}(i)} \right| \geq \epsilon \right\} \right| = 0 \text{ for each } m \text{ and } \epsilon > 0.$$

Proof. The definition of asymptotically statistically equivalent of multiple L can be interpreted as the following:

$$\left| \frac{x_s}{y_s} - L \right| \leq \epsilon \text{ for almost all } s(\text{ denoted by a.a.s}).$$

This implies that

$$(L - \epsilon) y_s \leq x_s \leq (L + \epsilon) y_s \text{ a.a.s.} \tag{2.1}$$

Let us consider the for all i, $R_n(\mathbf{B}_i x) = \sum_{p=n}^{\infty} \sum_{r=1}^{\infty} b_{pr(i)} x_r$; which implies the following

$$\begin{aligned} \frac{R_n(\mathbf{B}_i x)}{R_n(\mathbf{B}_i y)} &\leq \frac{\sum_{r=1}^{R-1} \sum_{p=n}^{\infty} \max_{0 \leq r \leq R-1} \{b_{p,r}(i)\} x_r}{\sum_{p=n}^{\infty} \sum_{r=1}^{\infty} b_{p,r}(i) y_r} \\ &+ \frac{\sum_{p=n}^{\infty} \sum_{r=R}^{\infty} b_{p,r}(i) x_r}{\sum_{p=n}^{\infty} \sum_{r=1}^{\infty} b_{p,r}(i) y_r}. \end{aligned}$$

Using (2.1) we obtain the following:

$$\frac{R_n(\mathbf{B}_i x)}{R_n(\mathbf{B}_i y)} \leq \frac{\sum_{r=1}^{R-1} \sum_{p=n}^{\infty} \max_{0 \leq r \leq R-1} \{b_{p,r}(i)\}}{\delta \sum_{p=k}^{\infty} \sum_{r=1}^{\infty} b_{p,r}(i)} + (L + \epsilon) \text{ a.a.n}$$

Thus by Equation (2) for all i

$$\limsup_n \frac{R_n(\mathbf{B}_i x)}{R_n(\mathbf{B}_i y)} \leq (L + \epsilon) \text{ a.a.n .}$$

Inequality (2.1) can be used in a similar manner to obtain the following:

$$\liminf_n \frac{R_n(\mathbf{B}_i x)}{R_n(\mathbf{B}_i y)} \geq (L - \epsilon) \text{ a.a.n .}$$

Thus

$$R_n(\mathbf{B}_i x) \overset{S_L}{\sim} R_n(\mathbf{B}_i y).$$

For the second part of this theorem let us consider the following two sequences:

$$x_s = \begin{cases} 0 & \text{if } s \leq K \\ 1 & \text{otherwise} \end{cases}$$

where K is a positive integer and $y_s = 1$ for all s . The two sequences imply the following: for all i,

$$\begin{aligned} R_n(\mathbf{B}_i x) = \sum_{k=n}^{\infty}(\mathbf{B}_i x) &= \sum_{k=n}^{\infty}\sum_{s=K+1}^{\infty} b_{k,s}(i) \\ &= \sum_{k=n}^{\infty}\sum_{s=1}^{\infty} b_{k,s}(i) - \sum_{k=n}^{\infty}\sum_{s=0}^{K} b_{k,s}(i). \end{aligned}$$

Therefore for all i

$$st - \liminf_n \frac{R_n(\mathbf{B}_i x)}{R_n(\mathbf{B}_i y)} \leq 1 - st - \limsup_n \frac{(K+1)\sum_{p=n}^{\infty} b_{p,k}(i)}{\sum_{p=n}^{\infty}\sum_{s=1}^{\infty} b_{p,s}(i)}$$

where $0 \leq k \leq K$. Since each nonconstant element of the last inequality has statistical limit zero we obtain the following for all i:

$$\lim_n \frac{R_n(\mathbf{B}_i x)}{R_n(\mathbf{B}_i y)} = 1 \text{ a.a.n } .$$

This completes the proof. □

In 1980 Pobyvanets presented definition for asymptotically equivalent sequences and asymptotic regular matrices. Using these definitions he also presented a Silverman Toeplitz type characterization for asymptotic equivalent sequences. In similar manner we have presented a definition for asymptotically statistical equivalent sequences via a sequence of infinite matrices and use it to present Silverman Toeplitz conditions similar to Poyvanents' results.

Theorem 2.2. *In order for a sequence of summability matrices $\boldsymbol{B}$ to be asymptotically statistical regular it is necessary and sufficient that for each fixed positive integer k_0*

(1) $\sum_{p=1}^{k_0} b_{n,p}(i)$ *is bounded for each (n,i).*
(2)

$$\lim_n \frac{1}{n}\max_i\left\{\textit{the number of } k \leq n : \left|\frac{\sum_{p=1}^{k_0} b_{n,p}(i)}{\sum_{p=k1}^{\infty} b_{n,p}(i)}\right| \geq \epsilon \textit{ for each } k_0 \textit{ and } \epsilon > 0\right\} = 0.$$

Proof. The necessary part of this theorem is established in a manner similar to that of the necessary part of the last theorem. To establish the sufficient part of this theorem, let $\epsilon > 0$ and $x \overset{S^L}{\sim} y$, $x \in P_0$ and $y \in P_\delta$ for some $\delta > 0$ these conditions implies that

$$(L-\epsilon)y_{s+\alpha} \leq x_{s+\alpha} \leq (L+\epsilon)y_{s+\alpha} \text{ a.a.s for some } \alpha = 1,2,3,\ldots. \tag{2.2}$$

Let us consider the following:

$$\begin{aligned}\frac{(\mathbf{B}_i x)_n}{(\mathbf{B}_i y)_n} &= \frac{\sum_{p=1}^{\alpha} b_{n,p}(i)x_p + \sum_{p=1+\alpha}^{\infty} b_{n,p}(i)x_p}{\sum_{p=1}^{\alpha} b_{n,p}(i)y_p + \sum_{p=1+\alpha}^{\infty} b_{n,p}(i)y_p} \\ &= \frac{\frac{\sum_{p=1}^{\alpha} b_{n,p}(i)x_p}{\sum_{p=1+\alpha}^{\infty} b_{n,p}(i)y_p} + \frac{\sum_{p=\alpha+1}^{\infty} b_{n,p}(i)x_p}{\sum_{p=\alpha+1}^{\infty} b_{n,p}(i)y_p}}{\frac{\sum_{p=1}^{\alpha} b_{n,p}(i)y_p}{\sum_{p=1+\alpha}^{\infty} b_{n,p}(i)y_p} + 1}.\end{aligned}$$

Inequality (2.2) implies that for all i,

$$\lim_n \frac{\sum_{p=1+\alpha}^{\infty} b_{n,p}(i)x_p}{\sum_{p=1+\alpha}^{\infty} b_{n,p}(i)y_p} = L, \text{ a.a.n.}$$

Since $x \in P_0$, $y \in P_\delta$, and condition (2) holds we obtain the following for all i

$$\lim_n \frac{\sum_{p=1}^{\alpha} b_{n,p}(i)x_p}{\sum_{p=1+\alpha}^{\infty} b_{n,p}(i)y_p} = 0, \text{ a.a.n}$$

and

$$\lim_n \frac{\sum_{p=1}^{\alpha} b_{n,p}(i)y_p}{\sum_{p=1+\alpha}^{\infty} b_{n,p}(i)y_p} = 0, \text{ a.a.n.}$$

Thus for all i

$$\lim_n \frac{(\mathbf{B}_i x)_n}{(\mathbf{B}_i y)_n} = L, \text{ a.a.n.}$$

This implies that $Bx \overset{S^L}{\sim} By$ where $x \overset{S^L}{\sim} y$, $y \in P_0$, and $y \in P_\delta$ for some $\delta > 0$. This completes the proof. □

If we let $B_i = A$ for all i the above theorems reduces to Patterson's results in [6]

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Ekrem Savaş: Yüzüncü Yıl University, Department of Mathematics Van, TURKEY
E-mail: ekremsavas@yahoo.com

Richard F. Patterson: Department of Mathematics and Statistics, University of North Florida Jacksonville, Florida, 32224, USA
E-mail: rpatters@unf.edu