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TIM STARR, TREVOR WEST AND A POSITIVE CONTRIBUTION TO OPERATOR THEORY

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We have known Trevor West since before he was Trevor West, and that's not yesterday. Our first encounter would have been over fifty years ago, when we both presented for the Entrance Scholarship Examination for Trinity College. And he knew all about us: the Midleton Mafia and Bennet's Brewery Boys [17] had us sussed. But they didn't know about The Plough: "Twixt finger and thumb, the squat pen sits: I'll dig with it." Seamus Heaney [10] knew all about The Plough - it got him a Nobel Prize. And dig with it we did - all through the physics paper - while the young Tim Starr sat and chewed the end of his pencil (they were very poor in Midleton in those days). This must have disconcerted him, because shortly afterwards Tim Starr changed his name to Trevor West. Then of course he rallied himself, soon scaling the heights of college society. By the time we came out the other end of Trinity College he had appointed himself our political commissar, vigilant lest we should reveal to the waiting Brits the depths of our collective ignorance of matters Tripos Three. He fumbled the ball a little on his first outing when he had to be brought back to the Smythies seminar with a proper proof of Lemma 5 [11]:

Hans Freudenthal single handedly kept West and ourself away from partially ordered vector spaces for nearly twenty years. But then he was handed the very cold hot potato that was the Olangunju thesis, which together with two research

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papers he had gone back to Nigeria to rewrite. And Shell Oil got him: he was killed in a motor accident. So this stuff was orbiting Mill Lane, full of little notes ("Dear John, what now? Yours, Frank"). "What now" turned out to be Trevor West, who rose to the occasion. If the late P.A. Olagunju taught him the art of ghost writing [12], then he taught himself the art of spin, converting a simple error in the theory of bounded operators on incomplete normed spaces into a definition [15]. Trevor West has been an operator with single spectrum; this is not his obituary, but we do have [16] his epitaph: "here lies the West decomposition".

Late in life Trevor West has, backed by the resources of the Northern Ireland Civil Service, turned his thoughts towards that old comedy duo Perron and Frobenius, who tell us [13],[14], that under certain circumstances the spectral radius of a positive operator belongs to its spectrum. This holds for positive matrices, and more generally for positive operators on well behaved Banach lattices, and also for positive definite operators on Hilbert spaces. On the day that is in it we would here like to put in front of you a Banach algebra setting for this, 99% due to Heinrich Raubenheimer and his student Sonja Rode, who went and married another of his students Henri du T. Mouton, which gives the bibliography here an astringent flavour [8],[9].

A complex Banach algebra A, with identity 1, is classified by Raubenheimer, Rode and Mouton as an *ordered Banach algebra* if [8],[9] there is distingushed a *positive cone* $A^+ \subseteq A$ which is closed under non negative real linear combinations and norm limits, contains the identity, and is also closed under multiplication: the product of positive elements should again be positive. This is fine for positive matrices, positive operators on vector lattices, and completely positive operators on C^* algebras, but not inside C^* algebras thenselves: the only way the product of two C^* positives can again be positive is that they commute. Thus our 1% contribution to this circle of ideas is to relax the Raubenheimer/Rode/Mouton concept of an "ordered Banach algebra":

1. Definition *A partially ordered Banach algebra is a complex Banach algebra A, with identity 1, which has a positive cone $A^+ \subseteq A$ subject to the following two (four?) conditions:*

(1.1) $$\boldsymbol{R}^+ A^+ + \boldsymbol{R}^+ A^+ \subseteq A^+ = cl\ A^+;$$

1.2 $$1 \in A^+ \bullet_{comm} A^+ \subseteq A^+,$$

where we write

1.3 $$H \bullet_{comm} K = \{a \cdot b : a \in H, b \in K, b \cdot a = a \cdot b\}$$

for the "commuting product" of subsets H, K of A.

Note that it is part of our concept of "cone" that

1.4 $$A^+ \cap (-A^+) = \{0\}.$$

One source of positive cones would [3],[4] be a "modulus function" $|\cdot| : A \to A$ with natural properties, giving

1.5 $$A^+ = \{|a| : a \in A\} = \{a \in A : a = |a|\}\ .$$

Definition 1 captures all the "positive" operators discussed by Raubenheimer/Rode/Mouton, as well as C^* algebras, but not ([1] Theorem 5.8) the numerical-range positivity concept in general Banach algebras.

To obtain a "Perron-Frobenius theorem" Raubenheimer/Rode/Mouton call on a certain "spectral radius monotonicity": if a and b and $b-a$ are all positive then the spectral radius of a should be less than or equal to that of b. This they can derive from a "normality" property of the cone A^+: it is appropriate here to also relax this assumption. Thus we shall call the cone A^+ *commutatively normal* if there is $k > 0$ for which there is implication

1.6 $$ba = ab\ ,\ 0 \le a \le b \Longrightarrow \|a\| \le k\|b\|.$$

where of course we write

1.7 $$a \le b \Longleftrightarrow b - a \in A^+ \Longleftrightarrow b \ge a.$$

A partially ordered Banach algebra with a commutatively normal cone has "commuting spectral radius monotonicity":

2. Theorem *If A is a partially ordered Banach algebra with a commutatively normal cone then whenever*

2.1 $$0 \le a \le b \textit{ with } ab = ba$$

we have

2.2 $$|a|_\sigma \le |b|_\sigma \ .$$

Proof. Simply notice

2.3 $$0 \le a^n \le a^{n-1}b \le \ldots \le ab^{n-1} \le b^n \ ,$$

and hence

2.4 $$\|a^n\| \le k\|b^n\| \ (n \in \mathbb{N}) \ .$$

Now

$$|a|_\sigma = \lim_n \|a^n\|^{1/n} \le \lim_n k^{1/n}\|b^n\|^{1/n} = |b|_\sigma \ \bullet$$

Raubenheimer/Rode/Mouton [8],[9] obtain (2.2) whenever $0 \le a \le b$, but only because of their more restrictive concept of positive cone: they assume

2.5 $$A^+ \cdot A^+ \subseteq A^+,$$

without requiring commutivity. This misses the important case of a C^*-algebra, although Theorem 1 is trivial there. Armed with spectral radius monotonicity, Raubenheimer/Rode/Mouton are able ([8] Theorem 5.2) to deduce the Perron-Frobenius theorem; we observe that their proof goes through in our marginally more relaxed environment. The argument proceeds by contradiction, and uses one tiny bit of actual mathematics - a sort of dry run for Stirling's formula:

3. Theorem *If A is a partially ordered Banach algebra with commuting spectral radius monotonicity then there is implication*

3.1 $$0 \le a \Longrightarrow |a|_\sigma \in \sigma(a).$$

Proof. Recall the simple inequality

3.2 $$(n-1)! \le n^n e^{1-n} \le n! \ ,$$

obtained by integrating ([5] Ch XII §1 Theorem 1) the log function between 1 and n and looking at upper and lower Riemann sums. For the contradiction suppose

3.3 $$|a|_\sigma \notin \sigma(a) :$$

this immediately precludes the case $|a|_\sigma = 0$, where without any positivity the spectral radius would actually be the spectrum. If (3.3) holds then there must be $0 < \alpha < 1$ for which

3.4 $$\sigma(a) \subseteq \{\text{Re } z \le \alpha |a|_\sigma\}.$$

If $t > 0$ is arbitrary then the spectral mapping theorem says

$$\sigma(e^{ta}) \subseteq \{|z| \le e^{t\alpha|a|_\sigma}\} \ .$$

Since the cone A^+ is closed we have

$$0 \le \frac{t^n}{n!} \le \sum_{j=0}^{n+m} \frac{t^j}{j!} \le e^{t\alpha a} \ ,$$

and hence by spectral radius monotonicity

3.5 $$0 \le \frac{t^n}{n!}|a|_\sigma^n = |\frac{t^n}{n!}a^n|_\sigma \le |e^{ta}|_\sigma \le e^{t\alpha|a|_\sigma}.$$

This holds for all $t > 0$, in particular for $t = n/\alpha|a|_\sigma$, giving

3.6 $$\frac{n^n}{n!} \le \alpha^n e^n,$$

which for sufficiently large n contradicts the first part of (3.2). •

The argument for Theorem 3 is based on de Pagter and Schep ([7] Proposition 3.3), for positive operators on Banach lattices. If in particular the spectral radius is a pole of the resolvent, therefore also an eigenvalue, then an eigenvector $x \in A$ for the left multiplication L_c induced by $c = |a|_\sigma - a$ can be built from the Laurent expansion of $(c-z)^{-1}$, and will not only commute with $a \in A$ but also be a positive element: $x \in A^+$. This seems to hint at a sort of "positive point spectrum" for $a \in A$, which would be a subset of the usual point spectrum. More generally, without assuming isolation, the spectral radius of $a \in A^+$ will necessarily lie in the topological boundary of the spectrum of a, and hence the approximate point spectrum. Here again the approximate eigenvectors x_n can be made to commute with a, and also taken to be positive. Specifically take

3.7 $$x_n = u_t/\|u_t\| \text{ with } u_t = (t-a)^{-1} \text{ with } t = |a|_\sigma + 1/n \ :$$

note that the Neumann series converges and the partial sums are positive;

I am grateful to Sonja Rode/ Mouton for pointing out that it is entirely irrelevant whether or not

3.8 $$a \le |a|_\sigma \ .$$

At any rate the approximate point spectrum of $a \in A^+$ seems also to contain as a subset a "positive approximate point spectrum". At the other extreme the spectrum of $a \in A^+$ should be included in a larger "positive spectrum", from which $0 \in \mathbb{C}$ is to be excluded provided $a \in A^+$ has not just an inverse but a positive inverse $a^{-1} \in A^+$. There would then be clear blue water between positive matrices and C^*-algebra elements: in a C^*-algebra every positive element which is invertible automatically has a positive inverse, while only very special positive matrices can have positive inverses. Specifically, a positive matrix with a positive inverse has to be the product (in either order, but not commutatively) of a diagonal and a permutation.

This narrative rather tails off here, and is offered as a sort of literary version of the *Bord na Mona* medal, awarded [18] to Tim Starr and Trevor West, "for Positive Contributions to Operator Theory".

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