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UPPER TRIANGULAR OPERATORS WITH SVEP: SPECTRAL PROPERTIES

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Abstract

Spectral properties of upper triangular operators $T = (T_{ij})_{1 \leq i,j \leq n} \in B(\mathcal{X}^n)$, where $\mathcal{X}^n = \oplus_{i=1}^n \mathcal{X}_i$ and $\mathcal{X}_i$ is an infinite dimensional complex Banach space, such that $T_{ii} - \lambda$ has the single-valued extension property, SVEP, for all complex λ are studied.

1. Introduction

Let $B(\mathcal{X}^n)$ denote the algebra of operators (equivalently, bounded linear transformations) on the Banach space $\mathcal{X}^n = \oplus_{i=1}^n \mathcal{X}_i$, where $\mathcal{X}_i$, for all $1 \leq i \leq n$, is an infinite dimensional complex Banach space. A block matrix operator $T = (T_{ij})_{1 \leq i,j \leq n} \in B(\mathcal{X}^n)$ is upper triangular if T_{ij} is the 0 operator for all $j > i$. The spectral properties of upper triangular operators have been studied by a number of authors in the recent past (see [3, 4], [6], [8] and [13] for further references). If we define the diagonal operator $N \in B(\mathcal{X}^n)$ by $N = \oplus_{i=1}^n T_{ii}$, then it is apparent that the upper triangular operator $T = (T_{ij})_{1 \leq i,j \leq n}$ is the sum of N with an n-nilpotent operator Q (which,

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one would expect in general, does not commute with N). Consequently, the relationship between the spectra, and their various distinguished parts, of T and N is unlikely to be a straightforward one. What is of interest here is the determination of classes of operators T_{ii} such that T and N have as close a *spectral picture* as reasonably possible. A (separable) Hilbert space operator A, $A \in B(\mathcal{H})$, is a Jordan operator of order n, denoted $A \in \mathcal{J}_n(\mathcal{H})$, if $A = N + Q$, where Q is an n-nilpotent operator which commutes with the normal operator N; $A \in B(\mathcal{H})$ is an n-normal operator, denoted $A \in \mathcal{N}_n(\mathcal{H})$, if there exist a Hilbert space $\mathcal{K}$ and a unitary isomorphism $\Phi(: \mathcal{H} \longrightarrow \mathcal{K}^n = \mathcal{K} \oplus ... \oplus \mathcal{K}$, n copies) such that $\Phi A \Phi^{-1} = (N_{ij})_{1 \leq i,j \leq n} \in B(\mathcal{K}^n)$, where N_{ij} are mutually commuting normal operators; and $A \in B(\mathcal{H})$ is a $\mathcal{C}_n(\mathcal{H})$ operator if there exists a decomposition $\mathcal{H} = \oplus_{i=1}^n \mathcal{H}_i$ such that $A = (A_{ij})_{1 \leq i,j \leq n}$, $A_{ij} \in B(\mathcal{H}_j, \mathcal{H}_i)$, $A_{ij} = 0$ for $i > j$ and A_{ii} is a normal operator for all $1 \leq i \leq n$. It is not difficult to verify that $\mathcal{J}_n(\mathcal{H}) \cup \mathcal{N}_n(\mathcal{H}) \subset \mathcal{C}_n(\mathcal{H})$ [13, Proposition 1.4]. Upper triangular operators $\mathcal{C}_n(\mathcal{H})$ have been considered by Jung, Ko and Pearcy [13], who have shown that these operators share a number of properties with normal operators (see [13, Theorem 2.3 and Lemma 2.4]). For a normal operator A, both A and its adjoint A^* have the single-valued extension property, SVEP for short. Upper triangular operators $T \in B(\mathcal{X}^n)$ such that both T_{ii} and its conjugate T_{ii}^* have SVEP for all $1 \leq i \leq n$ have been considered by Benhida, Zerouali and Zguitti [3], who have shown that results similar to those for operators in $\mathcal{C}_n(\mathcal{H})$ hold for such operators. This paper extends, and adds to, the results of [3] by separately considering the cases T_{ii} has SVEP and T_{ii}^* has SVEP. With notation as explained below, we prove the following. "If $T \in B(\mathcal{X}^n)$ is an upper triangular operator such that T_i $(= T_{ii})$ and T_i^* have SVEP for all $1 \leq i \leq n$, then (i) $\sigma(T_i) = \sigma_s(T_i) = \sigma_a(T_i)$, $1 \leq i \leq n$, and $\sigma(T) = \sigma_s(T) = \sigma_a(T) = \cup_{i=1}^n \sigma(T_i)$; (ii) $\sigma_e(T) = \sigma_{sF}(T) = \mathrm{acc}\sigma_a(T) \cup \{\lambda \in \mathrm{iso}\sigma_a(T) : \dim H_0(T-\lambda) = \infty\}$; and (iii) $\sigma_b(T_i) = \sigma_{ba}(T_i) = \sigma_{bs}(T_i) = \sigma_w(T_i) = \sigma_{ws}(T_i) = \sigma_{wa}(T_i) = \sigma(T_i) \setminus \pi_0(T_i)$, $1 \leq i \leq n$, and $\sigma_b(T) = \sigma_{ba}(T) = \sigma_{bs}(T) = \sigma_w(T) = \sigma_{ws}(T) = \sigma_{wa}(T) = \cup_{i=1}^n \sigma_b(T_i) = \sigma(T) \setminus \pi_0(T)$. Furthermore, if $Q \in B(\mathcal{X}^n)$ is a quasinilpotent operator which commutes with T, then (iv) $\sigma(T+Q) = \sigma_a(T+Q)$, $\sigma_b(T+Q) = \sigma_{ba}(T+Q) = \sigma_{bs}(T+Q) = \sigma_w(T+Q) = \sigma_{wa}(T+Q) = \sigma_{ws}(T+Q) = \sigma_b(T)$ and $\sigma_{sF}(T+Q) = \sigma_e(T)$." For upper triangular operators T such that T_i has SVEP and the quasinilpotent part $H_0(T_i - \lambda I) = (T_i - \lambda I)^{-p_i}(0)$ for some integer $p_i \geq 1$ at points $\lambda \in \mathrm{iso}\sigma(T_i)$ (all $1 \leq i \leq n$), we prove that $\sigma_a(T^*) \setminus \sigma_{wa}(T^*) = \pi_{00}^a(T^*)$, and $\sigma(f(B)) \setminus \sigma_w(f(B)) = \pi_{00}(f(B))$, $B = T$ or T^*, for every non-constant function f which is analytic on a neighbourhood of $\sigma(T)$.

2. Notation and terminology

We shall henceforth shorten $A - \lambda I$ to $A - \lambda$. $\mathcal{X}$ and $\mathcal{X}_i$ $(1 \leq i \leq n)$ shall denote (infinite dimensional complex) Banach spaces, and $\mathcal{H}$ and $\mathcal{H}_i$ $(1 \leq i \leq n)$ shall denote separable (infinite dimensional complex) Hilbert spaces. As above, we shall define $\mathcal{X}^n$ by $\mathcal{X}^n = \oplus_{i=1}^n \mathcal{X}_i$ and $\mathcal{H}^n$ by $\mathcal{H}^n = \oplus_{i=1}^n \mathcal{H}_i$. A Banach space operator A, $A \in B(\mathcal{X})$, is said to be *left semi-Fredholm* (resp. *right semi-Fredholm*), $A \in \rho_{lF}(\mathcal{X})$ (resp., $A \in \rho_{rF}(\mathcal{X})$), if $A\mathcal{X}$ is closed and the deficiency index $\alpha(A) = \dim(A^{-1}(0))$ is finite (resp., the deficiency index $\beta(A) = \dim(\mathcal{X} \setminus A\mathcal{X})$ is finite); A is semi-Fredholm, $A \in \rho_{sF}(\mathcal{X})$, if $A \in \rho_{lF}(\mathcal{X}) \cup \rho_{rF}(\mathcal{X})$, and A is Fredholm, $A \in \rho_F(\mathcal{X})$, if $A \in \rho_{lF}(\mathcal{X}) \cap \rho_{rF}(\mathcal{X})$. The semi-Fredholm index of A, $\operatorname{ind}(A)$, is the (finite or infinite) number $\operatorname{ind}(A) = \alpha(A) - \beta(A)$. The left semi-Fredholm spectrum, the right semi-Fredholm spectrum and the Fredholm spectrum of A are, respectively, the sets $\sigma_{le}(A) = \{\lambda : A - \lambda \notin \rho_{lF}(\mathcal{X})\}$, $\sigma_{re}(A) = \{\lambda : A - \lambda \notin \rho_{rF}(\mathcal{X})\}$ and $\sigma_e(A) = \{\lambda : A - \lambda \notin \rho_F(\mathcal{X})\}$. The *spectral picture* $\mathcal{SP}(A)$ of A consists of the (Fredholm) essential spectrum $\sigma_e(A)$, the holes and pseudoholes of $\sigma_e(A)$, and the indices associated with these holes and pseudoholes [14]. Recall that similar operators have the same spectral picture. We say that the operator A is *Weyl* if it is Fredholm of index 0. The *Weyl spectrum* $\sigma_w(A)$ of A is the set $\{\lambda : A - \lambda \text{ is not Weyl}\}$. Let $\sigma(A)$, $\sigma_a(A)$, $\sigma_s(A)$, $\operatorname{acc}\sigma(A)$, $\operatorname{iso}\sigma(A)$, $\pi_0(A)$ and $\pi_{00}(A)$ denote, respectively, the spectrum, the approximate point spectrum, the approximate defect (or, surjectivity) spectrum, the accumulation points of the spectrum, the isolated points of the spectrum, the Riesz points, and the isolated eigenvalues of finite multiplicity of A. Let $\pi_0^a(A) = \pi_0(A) \cap \sigma_a(A)$ and $\pi_{00}^a(A) = \pi_{00}(A) \cap \sigma_a(A)$. In keeping with current terminology, we say that A *satisfies Weyl's theorem* if it satisfies the Weyl condition $\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A)$. (We refer the interested reader to [1, Chapter 3.8] for an excellent account of Browder and Weyl theorems.)

The *ascent of* $A - \lambda$, $\operatorname{asc}(A - \lambda)$, is the least non-negative integer n such that $(A - \lambda)^{-n}(0) = (A - \lambda)^{-(n+1)}(0)$; the *descent of* $(A - \lambda)$, $\operatorname{dsc}(A - \lambda)$, is the least non-negative integer n such that $(A - \lambda)^n \mathcal{X} = (A - \lambda)^{n+1} \mathcal{X}$. We say that A *has finite ascent (finite descent) if* $A - \lambda$ *has finite ascent (resp., descent) for all* λ. Recall from [1, Theorem 3.4] that if $\operatorname{asc}(A) < \infty$, then $\alpha(A) \leq \beta(A)$. The *Browder spectrum* $\sigma_b(A)$ of an operator A is the set of λ such that $A - \lambda$ is not Fredholm of finite ascent and descent. The *essential approximate point spectrum*, the *Browder essential approximate point spectrum*, the *essential defect spectrum* and the *Browder essential defect spectrum* of A are, respectively, the sets $\sigma_{wa}(A) = \{\lambda : A - \lambda \notin \rho_{lF}(\mathcal{X})$ or ind$(A-\lambda) \not\leq 0\}$, $\sigma_{ba}(A) = \{\lambda : A-\lambda \notin \rho_{lF}(\mathcal{X})$ or asc$(A-\lambda) = \infty\}$, $\sigma_{ws}(A) = \{\lambda : A-\lambda \notin \rho_{rF}(\mathcal{X})$ or ind$(A-\lambda \not\geq 0\}$ and $\sigma_{bs}(A)) = \{\lambda : A-\lambda \notin \rho_{rF}(\mathcal{X})$ or dsc$(A-\lambda) = \infty\}$. Apparently, $\sigma_{wa}(A) \subseteq \sigma_{ba}(A)$, $\sigma_{wa}(A) = \sigma_{ws}(A^*)$ and $\sigma_{ba}(A) = \sigma_{bs}(A^*)$.

$A \in B(\mathcal{X})$ has *the single-valued extension property* at λ_0, SVEP at λ_0 for short, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \to \mathcal{X}$ which satisfies

$$(A-\lambda)f(\lambda) = 0 \quad \text{for all} \quad \lambda \in \mathcal{D}_{\lambda_0}$$

is the function $f \equiv 0$. Trivially, every operator A has SVEP at points of the resolvent $\rho(A) = \mathbf{C} \setminus \sigma(A)$ and at points $\lambda \in$ iso$\sigma(A)$. *We say that A has SVEP if it has SVEP at every complex number λ.* The *quasinilpotent part* $H_0(A-\lambda)$ and the *analytic core* $K(A-\lambda)$ of $(A-\lambda)$ are defined by

$$H_0(A-\lambda) = \{x \in \mathcal{X} : \lim_{n \longrightarrow \infty} ||(A-\lambda)^n x||^{\frac{1}{n}} = 0\}$$

and

$$\begin{aligned} K(A-\lambda) = \{x \in \mathcal{X} : & \text{ there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \\ & \text{for which } x = x_0, (A-\lambda)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all} \\ & n = 1, 2, ...\}. \end{aligned}$$

We note that $H_0(A-\lambda)$ and $K(A-\lambda)$ are (generally) non-closed hyperinvariant subspaces of $(A-\lambda)$ such that $(A-\lambda)^{-m}(0) \subseteq H_0(A-\lambda)$ for all $m = 0, 1, 2, ...$ and $(A-\lambda)K(A-\lambda) = K(A-\lambda)$. We say that an operator $A \in B(\mathcal{X})$ satisfies *property* $H(p)$ for some integer $p \geq 1$ if $H_0(A-\lambda) = (A-\lambda)^{-p}(0)$ for all complex λ. (The interested reader is referred to [1] for these, and other, results on "local spectral theory".)

3. Spectral properties

Henceforth, $\mathcal{T}(\mathcal{X}^n)$ shall denote the class of upper triangular operators in $B(\mathcal{X}^n)$, and $T = (T_{ij})_{1 \leq i,j \leq n}$ shall denote an element of $\mathcal{T}(\mathcal{X}^n)$ such that the elements $T_i = T_{ii} \in B(\mathcal{X}_i)$, $1 \leq i \leq n$, along the main diagonal of T have SVEP. Given an operator A, $H(A)$ shall denote the set of (non-constant) functions f which are analytic on a neighbourhood of $\sigma(A)$.

Recall from [1, Theorem 2.9] that a sufficient condition for an operator $A = \begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix} \in B(\mathcal{X}^2)$ to have SVEP is that both A_1 and A_2 have SVEP. Applying a finitely repeated argument, it follows that:

Lemma 3.1 *T has SVEP.*

If an operator A has SVEP, then $\sigma(A) = \sigma_s(A)$ [1, Corollary 2.45]. The following lemma, which follows from a finitely repeated application of [6, Theorem 2.3], relates $\sigma(T)$ and $\sigma_e(T)$ to $\sigma(T_i)$ and $\sigma_e(T_i)$ (respectively).

Lemma 3.2 *$\sigma(T) = \cup_{i=1}^n \sigma(T_i)$ and $\sigma_e(T) = \cup_{i=1}^n \sigma_e(T_i)$.*

Remark here that either of the hypotheses "T_i has SVEP for $2 \leq i \leq n$" and "T_i^* has SVEP for $1 \leq i \leq n-1$" is sufficient for the equalities of Lemma 3.2 [6, Theorem 2.3]. The following lemma is probably known: we include it here for completeness.

Lemma 3.3 *$\sigma_b(T) = \cup_{i=1}^n \sigma_b(T_i)$.*

Proof. We prove the lemma for $A = \begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix}$, where A_1 and A_2 have SVEP; the proof for the general case follows from a finite induction argument. Evidently, $\sigma_b(A) \subseteq \sigma_b(A_1) \cup \sigma_b(A_2)$. Let $\lambda \notin \sigma_b(A)$. Then $A_1 - \lambda \in \rho_{lF}(\mathcal{X}_1)$ and $A_2 - \lambda \in \rho_{rF}(\mathcal{X}_2)$. Since A_i, $i = 1, 2$, has SVEP, $\text{ind}(A_i - \lambda) \leq 0$; again, since $A = \begin{pmatrix} I & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} I & * \\ 0 & I \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & I \end{pmatrix}$, the product index formula implies that $\text{ind}(A - \lambda) = \text{ind}(A_1 - \lambda) + \text{ind}(A_2 - \lambda) = 0$. Hence $\text{ind}(A_i - \lambda) = 0$ and $A_i - \lambda \in \rho_F(\mathcal{X})$ for $i = 1, 2$. But then, since A_i has SVEP and $\alpha(A_i - \lambda) = \beta(A_i - \lambda) < \infty$, $\text{asc}(A_i - \lambda) = \text{dsc}(A_i - \lambda) < \infty$ [1, Theorems 3.16, 3.4 and 3.74] $\Longrightarrow \lambda \notin \sigma_b(A_i)$; $i = 1, 2$. Hence $\cup_{i=1}^2 \sigma_b(A_i) \subseteq \sigma_b(A)$. □

Recall from [11, p.140] that an operator $A \in B(\mathcal{H}^n)$ is quasitriangular if there exists an increasing sequence $\{P_n\}$ of finite rank projections such that $P_n \longrightarrow I$ strongly as $n \longrightarrow \infty$ and $||(I - P_n)AP_n|| \longrightarrow 0$ (as $n \longrightarrow \infty$). Equivalently, A is quasitriangular if and only if $\mathcal{SP}(A)$ contains neither negative integers nor $-\infty$ (i.e., if and only if $\rho_{sF}^-(A) = \emptyset$, where $\rho_{sF}^-(A) = \{\lambda \in \rho_{sF}(A) : \text{ind}(A - \lambda) < 0\}$) [11, Theorem 6.4]). We say that A is *co-quasitriangular* if A^* is quasitriangular.

Lemma 3.4 *Operators $T \in B(\mathcal{H}^n)$ are co-quasitriangular.*

Proof. Since T has SVEP, $\lambda \in \rho_{sF}(T) \Longrightarrow \text{asc}(T - \lambda) < \infty$ [1, Theorem 3.16] $\Longrightarrow \text{ind}(T - \lambda) \leq 0 \Longrightarrow \rho_{sF}^-(T^*) = \emptyset$. □

The following proposition lists some of the more important spectral properties of operators in $\mathcal{T}(\mathcal{X}^n)$.

Proposition 3.5 *(i)* $\sigma(T_i) = \sigma_s(T_i)$, $1 \leq i \leq n$, *and* $\sigma(T) = \sigma_s(T) = \cup_{i=1}^n \sigma(T_i)$.

(ii) $\sigma_e(T_i) = \sigma_{re}(T_i)$, $1 \leq i \leq n$, *and* $\sigma_e(T) = \sigma_{re}(T) = \cup_{i=1}^n \sigma_e(T_i)$.

(iii) $\sigma_b(T_i) = \sigma_w(T_i) = \sigma_{ws}(T_i) = \sigma(T_i) \setminus \pi_0(T_i)$, $1 \leq i \leq n$, *and* $\sigma_b(T) = \sigma_w(T) = \sigma_{ws}(T) = \cup_{i=1}^n \sigma_b(T_i) = \sigma(T) \setminus \pi_0(T)$.

(iv) $\sigma_{wa}(T) = \sigma_{ba}(T) = \sigma_a(T) \setminus \pi_0^a(T)$.

(v) $\sigma_{sF}(T) = acc\sigma_a(T) \cup \{\lambda \in iso\sigma_a(T) : \dim H_0(T-\lambda) = \infty$ *or* $(T-\lambda)\mathcal{X}^n$ *is not closed*$\}$.

(vi) $f(\sigma_x(T)) = \sigma_x(f(T))$ *for every* $f \in H(T)$, *where* $\sigma_x = \sigma_w$ *or* σ_{wa}.

(vii) $T \in B(\mathcal{H}^n)$ *is bi-quasitriangular if and only if* T^* *has SVEP at points* $\lambda \notin \sigma_{sF}(T)$.

Proof. (i) has been proved above.

(ii). In view of Lemma 3.3, it will suffice to prove that if an operator $A \in B(\mathcal{X})$ has SVEP, then $\sigma_e(A) \subseteq \sigma_{re}(A)$. (Recall that the reverse inclusion holds for every operator A.) Consider a $\lambda \notin \sigma_{re}(A)$. Then $A - \lambda \in \rho_{rF}(\mathcal{X})$ with $\beta(A - \lambda) < \infty$. Since A has SVEP, $\operatorname{asc}(A - \lambda) < \infty$ [1, Theorem 3.16] $\Longrightarrow \operatorname{ind}(A - \lambda) \leq 0 \Longrightarrow \alpha(A - \lambda) \leq \beta(A - \lambda) < \infty \Longrightarrow \lambda \notin \sigma_e(A)$.

(iii) + (iv). Let $A \in B(\mathcal{X})$. Recall from [7, Lemma 2.18] that a necessary and sufficient condition for $\sigma_{wa}(A) = \sigma_{ba}(A) = \sigma_a(A) \setminus \pi_0^a(A)$ (similarly, $\sigma_w(A) = \sigma_b(A) = \sigma(A) \setminus \pi_0(A)$) is that A has SVEP at points $\lambda \notin \sigma_{wa}(A)$ (resp., $\lambda \notin \sigma_w(A)$). (In keeping with current terminology, we say that A satisfies a-Browder's theorem, respectively Browder's theorem, if $\sigma_{wa}(A) = \sigma_{ba}(A)$, respectively $\sigma_w(A) = \sigma_b(A)$: [7, Lemma 2.18] implies that operators A with SVEP satisfy a-Browder's, hence Browder's, theorem.) This proves (iv) and, in view of Lemmas 3.1 and 3.3, most of (iii). To complete the proof of (iii), we prove that $\sigma_w(A) = \sigma_{ws}(A)$ $(= \sigma_{wa}(A^*))$ for operators A with SVEP. Evidently, $\sigma_{ws}(A) \subseteq \sigma_w(A)$. Let $\lambda \notin \sigma_{ws}(A)$. Then $A - \lambda \in \rho_{rF}(\mathcal{X})$ and $\operatorname{ind}(A - \lambda) \geq 0$. Since A has SVEP implies $\operatorname{ind}(A - \lambda) \leq 0$, we conclude that $\operatorname{ind}(A - \lambda) = 0$ and $A - \lambda$ is Fredholm, i.e., $\lambda \notin \sigma_w(T)$.

(v). See [1, Theorem 3.79(i)].

(vi). Follows from an application of [16, Theorem 2] and [17, Theorem 1], since (SVEP $\Longrightarrow$) $\operatorname{ind}(T - \lambda) \leq 0$ for points $\lambda \in \rho_{sF}(T)$.

(vii). Let $\rho_{sF}^+(T) = \{\lambda \in \rho_{sF}(T) : \operatorname{ind}(T - \lambda) > 0\}$. Then T is bi-quasitriangular if and only if $\rho_{sF}^+(T) \cup \rho_{sF}^-(T) = \emptyset$. Evidently, T has SVEP implies $\rho_{sF}^+(T) = \emptyset$. Suppose that $\rho_{sF}^-(T) \neq \emptyset$. Then there exists a λ such that $T - \lambda$ is semi-Fredholm and $\operatorname{ind}(T - \lambda) \leq 0$. Since T^* has SVEP at $\overline{\lambda}$ (= the complex conjugate of λ), $\operatorname{dsc}(T - \lambda) < \infty$ [1, Theorem 3.17] $\Longrightarrow \operatorname{ind}(T - \lambda) \geq 0$. Hence $\operatorname{ind}(T - \lambda) = 0 \Longrightarrow \lambda \notin \rho_{sF}^-(T)$; contradiction. Conversely, assume that $\rho_{sF}^- \cup \rho_{sF}^+(T) = \emptyset$. Let $\lambda \notin \sigma_{sF}(T)$. Then $T - \lambda \in$

$\rho_{sF}(T)$ and $\mathrm{ind}(T-\lambda)=0 \Longrightarrow T-\lambda$ is Fredholm. Since T has SVEP at λ, $\mathrm{asc}(T-\lambda)<\infty$. Hence $\mathrm{asc}(T-\lambda)=\mathrm{dsc}(T-\lambda)<\infty$ [1, Theorem 3.4] $\Longrightarrow \lambda \in \mathrm{iso}\sigma(T)$ [1, Theorem 3.81] $\Longrightarrow T^*$ has SVEP at $\overline{\lambda}$. (It is evident from the argument above that if $A \in B(\mathcal{H}^n)$ has SVEP at points $\lambda \in \rho_{sF}(A)$, then a necessary and sufficient condition for A to be bi-quasitriangular is that A^* has SVEP at points $\lambda \in \rho_{sF}(A)$.) □

The following proposition is a (sort of) dual to Proposition 3.5.

Proposition 3.6 *Let $A=(A_{ij})_{1\le i,j\le n} \in B(\mathcal{X}^n)$ be an upper triangular operator such that $A_i^*=A_{ii}^*$ has SVEP for all $1\le i\le n$. Then:*

(i) $\sigma(A_i)=\sigma_a(A_i)$, $1\le i\le n$, and $\sigma(A)=\sigma_a(A)=\cup_{i=1}^n\sigma(A_i)$.

(ii) $\sigma_e(A_i)=\sigma_{le}(A_i)$, $1\le i\le n$, and $\sigma_e(A)=\sigma_{le}(A)=\cup_{i=1}^n\sigma_e(A_i)$.

(iii) $\sigma_b(A_i)=\sigma_w(A_i)=\sigma_{wa}(A_i)=\sigma(A_i)\setminus\pi_0(A_i)$, $1\le i\le n$, and $\sigma_b(A)=\sigma_w(A)=\sigma_{wa}(A)=\cup_{i=1}^n\sigma_b(A_i)=\sigma(A)\setminus\pi_0(A)$.

(iv) $\sigma_{ws}(A)=\sigma_{bs}(A)=\sigma_s(A)\setminus\pi_0^s(A)$.

(v) $\sigma_{sF}(A)=acc\sigma_s(A)\cup\{\lambda\in iso\sigma_s(A):\dim(\mathcal{X}^n\setminus K(A-\lambda))=\infty\}$.

(vi) $f(\sigma_x(A))=\sigma_x(f(A))$ for every $f\in H(A)$, where $\sigma_x=\sigma_w$ or σ_{ws}.

(vii) A is quasitriangular, if $\mathcal{X}_i=\mathcal{H}_i$, $1\le i\le n$.

Proof. Evidently, A^* has SVEP. Let $B=A_i$ or A; then $\mathrm{dsc}(B-\lambda)<\infty$ [1, Theorem 3.17] $\Longrightarrow \mathrm{ind}(B-\lambda)\ge 0$.

(i). If B^* has SVEP, then $\sigma(B)=\sigma_a(B)$ [1, Corollary 2.45].

(ii). Evidently, $\sigma_e(A)=\cup_{i=1}^n\sigma_e(A_i)$. Let $\lambda\notin\sigma_{le}(B)$; then $\alpha(B-\lambda)<\infty$ and $B-\lambda$ is semi-Fredholm $\Longrightarrow \mathrm{ind}(B-\lambda)\ge 0$ and $\alpha(B-\lambda)<\infty \Longrightarrow \beta(B-\lambda)\le\alpha(B-\lambda)<\infty \Longrightarrow \lambda\notin\sigma_e(T)$.

(iii)+ (iv). Argue as in the proof of Lemma 3.3 to prove that $\sigma_b(A)=\cup_{i=1}^n\sigma_b(A_i)$ (use the fact that $\mathrm{ind}(B-\lambda)\ge 0$). Observe that B^* satisfies a-Browder's theorem, so that $\sigma_b(B^*)=\sigma_w(B^*)=\sigma(B^*)\setminus\pi_0(B^*)$ and $\sigma_{wa}(B^*)=\sigma_{ba}(B^*)=\sigma_a(B^*)\setminus\pi_0^a(B^*)$. The proof follows, since $\sigma_x(B^*)=\sigma_x(B)$, where σ_x denotes σ or σ_b or σ_w, $\sigma_{wa}(B^*)=\sigma_{ws}(B)$, $\sigma_{ba}(B^*)=\sigma_{bs}(B)$, $\sigma_a(B^*)=\sigma_s(B)$ and $\pi_0^a(B^*)=\pi_0^s(B)$.

(v). See [1, Theorem 3.79(ii)].

(vi). Since $\mathrm{ind}(A-\lambda)\ge 0$ for points $\lambda\in\rho_{sF}(A)$, [16, Theorem 2] and [17, Theorem 1] apply.

(vii). $\rho_{sF}^-(A)=\emptyset$. □

Partial versions of the following theorem, which combines parts of Propositions 3.5 and 3.6, have been proved (for the more restrictive class of $\mathcal{C}_n(\mathcal{H}^n)$ of upper triangular operators with normal operator entries along the main diagonal) by Jung, Ko and Pearcy [13] and Benhida, Zerouali and Zguitti [3].

Theorem 3.7 *If $T \in \mathcal{T}(\mathcal{X}^n)$ is such that T_i^* $(= T_{ii}^*)$ has SVEP for all $1 \leq i \leq n$, then:*

(i) $\sigma(T_i) = \sigma_s(T_i) = \sigma_a(T_i)$, $1 \leq i \leq n$, and $\sigma(T) = \sigma_s(T) = \sigma_a(T) = \cup_{i=1}^n \sigma(T_i)$.

(ii) $\sigma_e(T) = \sigma_{sF}(T) = acc\sigma_a(T) \cup \{\lambda \in iso\sigma_a(T) : \dim H_0(T-\lambda) = \infty\}$.

(iii) $\sigma_b(T_i) = \sigma_{ba}(T_i) = \sigma_{bs}(T_i) = \sigma_w(T_i) = \sigma_{ws}(T_i) = \sigma_{wa}(T_i) = \sigma(T_i) \setminus \pi_0(T_i)$, $1 \leq i \leq n$, and $\sigma_b(T) = \sigma_{ba}(T) = \sigma_{bs}(T) = \sigma_w(T) = \sigma_{ws}(T) = \sigma_{wa}(T) = \cup_{i=1}^n \sigma_b(T_i) = \sigma(T) \setminus \pi_0(T)$.

Furthermore, if $Q \in B(\mathcal{X}^n)$ is a quasinilpotent operator which commutes with T, then:

(iv) $\sigma(T+Q) = \sigma_a(T+Q)$, $\sigma_b(T+Q) = \sigma_{ba}(T+Q) = \sigma_{bs}(T+Q) = \sigma_w(T+Q) = \sigma_{wa}(T+Q) = \sigma_{ws}(T+Q) = \sigma_b(T)$ and $\sigma_{sF}(T+Q) = \sigma_e(T)$.

Proof. (i), (ii) and (iii) are evident. Observe that both Q and Q^* have SVEP, and Q^* commutes with T^*. Applying [1, Corollary 2.12] we conclude that both $T+Q$ and T^*+Q^* have SVEP. The proof (now) of (iv) is an immediate consequence of (the above and) [15, Theorems 5 and 6] which state that "for an operator $A \in B(\mathcal{X})$, $\sigma_b(A)$ (resp., $\sigma_{ba}(A)$) is the largest subset of $\sigma(A)$ (resp., $\sigma_a(A)$) which remains invariant under perturbations by commuting Riesz operators". $\square$

Remark 3.8 It is easy to verify that $\sigma_a(T+Q) = \sigma_a(T)$ in the case in which Q is nilpotent. If one assumes in Theorem 3.7(iv) that the quasinilpotent operator Q is injective, then (again) $\sigma(T+Q) = \sigma_a(T+Q) = \sigma_a(T)$. To see this we prove that if $Q \in B(\mathcal{X})$ is an injective quasinilpotent which commutes with $A \in B(\mathcal{X})$, then $\sigma_a(A+Q) = \sigma_a(A)$. Because of symmetry, it would suffice to prove that $\sigma_a(A+Q) \subseteq \sigma_a(A)$. If $\lambda \notin \sigma_a(A)$, then $\lambda \notin \sigma_{wa}(A) = \sigma_{wa}(T+Q)$ [15]. In particular, $T+Q-\lambda$ has closed range, and if $\alpha(T+Q-\lambda) = 0$, then $\lambda \notin \sigma_a(T+Q)$. Thus, we have to prove that $\alpha(T+Q-\lambda) = 0$. Evidently, $\alpha(T-\lambda) < \infty$. Let $x \in (T+Q-\lambda)^{-1}(0)$. Then $Q^m x \in (T+Q-\lambda)^{-1}(0)$ for all $m = 0, 1, 2, \ldots$. It is straightforward to see, using the injectivity of Q, that if $p(Q)x = 0$ for some polynomial $p(t) = \sum_{i=0}^m c_i t^{m-i}$ then $p(.)$ is identically 0. Hence the sequence of vectors $\{Q^m x\}$ is linearly independent. This is a contradiction.

Remark 3.9 A very large number of the important classes of operators in $B(\mathcal{X})$ have SVEP (hence elements belonging to these classes are suitable candidates for the choice of operators T_{ii} in Proposition 3.5). Recall that an operator $A \in B(\mathcal{H})$ is *hyponormal* if $|A^*|^2 \leq |A|^2$, *p-hyponormal* if $|A^*|^{2p} \leq |A|^{2p}$ for some $0 < p < 1$, *w-hyponormal* if $|\tilde{A}^*| \leq |A| \leq |\tilde{A}|$ (where, for U as in the polar decomposition $A = U|A|$ for A, $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ denotes the *first Aluthge transform* of A), *M-hyponormal* if $||(A-\lambda I)^*x|| \leq M||(A-\lambda I)x||$ for some scalar M, all $x \in \mathcal{H}$ and every complex number λ, *(p,k)-quasihyponormal* for $0 < p \leq 1$ and some integer $k \geq 1$ if $A^{*k}(|A|^{2p} - |A^*|^{2p})A^k \geq 0$, *totally $*$-paranormal* if $||(A-\lambda I)^*x||^2 \leq ||(A-\lambda I)^2x||$ for every unit vector x and complex number λ, and *paranormal* if $||Ax||^2 \leq ||A^2x||$ for all unit vectors $x \in \mathcal{H}$. (Evidently, a $(1,1)$-quasihyponormal operator is quasihyponormal and a $(1,k)$-quasihyponormal operator is k-quasihyponormal.) All these classes of (Hilbert space) operators have SVEP ,indeed more. (We refer the reader to the monograph [9] for information on these classes of operators; see also [1], [7] and [10].) If $\mathcal{X}$ is separable (resp., reflexive), then paranormal operators $A \in B(\mathcal{X})$ (resp., operators $A \in B(\mathcal{X})$ satisfying a *local growth condition of some order* $m \geq 1$, $A \in$ locally $-(G_m)$), have SVEP; see [5] and [12]. (See Section 3 for definition of (G_m) and locally $-(G_m)$ operators.) Operators satisfying property $H(p)$, for some integer $p \geq 1$, have SVEP. Class $H(p)$ is large; it contains, amongst other classes, the class of *generalized scalar operators* and *multipliers of commutative semi-simple Banach algebras* [1, pp. 170 - 176 and 215].

Remark 3.10 Both A and the conjugate operator A^* have SVEP for *decomposable*, in particular normal, operators A [1]: Theorem 3.7 holds for operators T such that T_{ii} is decomposable for all $1 \leq i \leq n$.

4. Operators $T \in \mathcal{T}(\mathcal{X}^n)$ satisfying $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$ and $\sigma_a(T) \setminus \sigma_{wa}(T) = \pi^a_{00}(T)$.

SVEP alone is not enough for an operator A to satisfy Weyl's theorem (i.e., $\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A)$): consider, for example, the operator $A = Q \oplus S$, where $Q \in B(\mathcal{H})$ is quasinilpotent and S is a nilpotent on a finite dimensional space, which satisfies $\sigma(A) = \sigma_w(A) = \{0\}$ and $\pi_{00}(A) = \{0\}$. For operators A satisfying property $H(p)$, $f(A)$ and $f(A^*)$ satisfy Weyl's theorem for every $f \in H(A)$ [1, Theorem 3.104]. A similar result holds for operators $T \in \mathcal{T}(\mathcal{X}^n)$ satisfying $H_0(T_i - \lambda) = (T_i - \lambda)^{-p_i}(0)$ for some integer $p_i \geq 1$ $(1 \leq i \leq n)$, as we shall momentarily see.

Let $A = \begin{pmatrix} A_1 & X \\ 0 & A_2 \end{pmatrix} \in B(\mathcal{X}^2)$, where A_1, A_2 have SVEP, and $H_0(A_i - \lambda) = (A_i - \lambda)^{-p_i}(0)$, $i = 1, 2$, for some integer $p_i \geq 1$ and complex number

λ. Let $x = x_1 \oplus x_2 \in \mathcal{X}^2$ be such that $x \in H_0(A-\lambda)$. Set $t_{1m} = (A_1 - \lambda)^m x_1 + \{\sum_{j=0}^{m-1} (A_1-\lambda)^{m-1-j} X (A_2-\lambda)^j\} x_2$ and $t_{2m} = (A_2-\lambda)^m x_2$. Then $(A-\lambda)^m x = t_{1m} \oplus t_{2m}$, and $\lim_{m\longrightarrow\infty} ||t_{im}||^{\frac{1}{m}} = 0$ $(i = 1, 2)$. Evidently, $t_{2p_2} = 0$ and

$$t_{1m} = (A_1-\lambda)^{p_1}[(A_1-\lambda)^{m-p_1}x_1 + \{\sum_{j=0}^{m-p_2-1} (A_1-\lambda)^{m-p_1-1-j} X (A_2-\lambda)^j\} x_2].$$

Since $H_0(A_1-\lambda) = (A_1-\lambda)^{-p_1}(0)$,

$$[(A_1-\lambda)^{m-p_1}x_1 + \{\sum_{j=0}^{m-p_2-1} (A_1-\lambda)^{m-p_1-1-j} X (A_2-\lambda)^j\} x_2] \in (A_1-\lambda)^{-p_1}(0).$$

Hence

$$H_0(A-\lambda) = \oplus_{i=1}^2 H_0(A_i-\lambda) = \oplus_{i=1}^2 (A_i-\lambda)^{-p_i}(0) = (A-\lambda)^{-p}(0),$$

where $p = \max\{p_1, p_2\}$. Repeating this argument a finite number of times it follows that if $A = (A_{ij})_{1\le i,j\le n} \in B(\mathcal{X}^n)$ is an upper triangular operator such that $H_0(A_{ii}-\lambda) = (A_{ii}-\lambda)^{p_i}$, $1 \le i \le n$, for some integer $p_i \ge 1$ and complex number λ, then

$$H_0(A-\lambda) = \oplus_{i=1}^n H_0(A_i-\lambda) = \oplus_{i=1}^n (A_i-\lambda)^{-p_i}(0) = (A-\lambda)^{-p}(0),$$

where $p = \max\{p_1, ..., p_n\}$.

Recall [1] that A satisfies a-Weyl's theorem if $\sigma_a(A) \setminus \sigma_{wa}(a) = \pi_{00}^a(A)$.

Theorem 4.1 *If $T \in \mathcal{T}(\mathcal{X}^n)$ is such that $H_0(T_i-\lambda) = (T_i-\lambda)^{-p_i}(0)$, $1 \le i \le n$, at points $\lambda \in iso\sigma(T_i)$, then (i) $f(T)$ and $f(T^*)$ satisfy Weyl's theorem (i.e., $\sigma(f(B)) \setminus \sigma_w(f(B)) = \pi_{00}(f(B))$) for every $f \in H(T)$, where $B = T$ or T^*) and (ii) T^* satisfies a-Weyl's theorem .*

Proof. We start by proving that T and T^* satisfy Weyl's theorem. Recall from Proposition 3.5 that $\sigma(T) \setminus \sigma_w(T) = \pi_0(T)$. Hence $\sigma(T) \setminus \sigma_w(T) \subset \pi_{00}(T)$. to prove the reverse inclusion, we start by observing that iso$\sigma(T) \subseteq \cup_{i=1}^niso\sigma(T_i)$. (This is proved for 2×2 operators in $\mathcal{T}(\mathcal{X}^n)$ in [8, Lemma 2.1]; the general case follows from a finite induction argument.) Let $\lambda \in \pi_{00}(T)$. Then $\lambda \in$ iso$\sigma(T_i) \cup \rho(T_i)$ for all $1 \le i \le n$. (Here $\rho(T_i)$ denote the resolvent set of T_i.) Following the convention that $H_0(T_i-\lambda) = (T_i-\lambda)^{-1}(0) = \{0\}$ for $\lambda \in \rho_F(T_i)$, we have from the above that

$$H_0(T-\lambda) = \oplus_{i=1}^n (T_i-\lambda)^{-p_i}(0) = (T-\lambda)^{-p}(0),$$

where $p = \max\{p_1, ..., p_n\}$. Hence, since $\lambda \in \mathrm{iso}\sigma(T)$,

$$\begin{aligned} \mathcal{X}^n &= H_0(T-\lambda) \oplus K(T-\lambda) = (T-\lambda)^{-p}(0) \oplus K(T-\lambda) \\ &\Longrightarrow (T-\lambda)^p \mathcal{X}^n = 0 \oplus (T-\lambda)^p K(T-\lambda) = K(T-\lambda) \\ &\Longrightarrow \mathcal{X}^n = (T-\lambda)^{-p}(0) \oplus (T-\lambda)^p \mathcal{X}^n, \end{aligned}$$

which implies that the isolated points of $\sigma(T)$ are poles of the resolvent of T ($\Longrightarrow T$ is *isoloid*, i.e. isolated points of $\sigma(T)$ are eigenvalues of T). Hence $\lambda \in \pi_{00}(T) \Longrightarrow \lambda \in \pi_0(T) \Longrightarrow \pi_{00}(T) \subseteq \pi_0(T)$, which implies that T satisfies Weyl's theorem. To prove that T^* satisfies Weyl's theorem, we start by observing that $\sigma(T) = \sigma(T^*)$, $\sigma_w(T) = \sigma_w(T^*)$ and $\pi_0(T) = \pi_0(T^*) \Longrightarrow \sigma(T^*)\backslash\sigma_w(T^*) = \pi_0(T^*) \subseteq \pi_{00}(T^*)$. Since $\lambda \in \pi_{00}(T^*) \Longrightarrow \lambda \in \mathrm{iso}\sigma(T^*) \Longrightarrow \lambda \in \mathrm{iso}\sigma(T) \Longrightarrow \lambda \in \pi_0(T) = \pi_0(T^*)$, $\pi_0(T^*) = \pi_{00}(T^*)$. Hence T^* satisfies Weyl's theorem.

Recall from Proposition 3.5 that $f(\sigma_w(T) = \sigma_w(f(T))$; hence, also, $f(\sigma_w(T^*) = \sigma_w(f(T^*))$. Since T is isoloid ($\Longrightarrow T^*$ is isoloid), [17, Theorem 4] implies that $f(T)$ and $f(T^*)$ satisfy Weyl's theorem. To complete the proof, we now show that T^* satisfies a-Weyl's theorem.

Recall from Proposition 3.5 that $\sigma(T^*) = \sigma_a(T^*)$ ($= \sigma_s(T)$) and $\sigma_w(T^*) = \sigma_{wa}(T^*)$ ($= \sigma_{ws}(T)$). Evidently, $\pi^a_{00}(T^*) = \pi_{00}(T^*)$; hence, since T^* satisfies Weyl's theorem (see above), $\sigma_a(T^*) \setminus \sigma_{wa}(T^*) = \pi^a_{00}(T^*)$. □

Theorem 4.1 applies, in particular, to operators $T \in \mathcal{T}(\mathcal{X}^n)$ such that each T_{ii} satisfies property $H(p_i)$. (Observe that property $H(p)$ implies finite ascent, hence SVEP.) It is known [7] that the isolated points of the spectrum of a paranormal operator are simple poles of the operator. Hence paranormal operators A satisfy property $H(1)$ at points $\lambda \in \mathrm{iso}\sigma(A)$. (Paranormal operators do not satisfy property $H(p)$ [2].)

An operator $A \in B(\mathcal{H})$ satisfies a growth condition of order m for some integer $m \geq 1$, $A \in (G_m)$, if there exists a scalar K such that $||(A-\lambda)^{-1}|| \leq \frac{K}{[\mathrm{dis}(\lambda,\sigma(A))]^m}$ for all $\lambda \notin \sigma(A)$. Apparently, $A \in (G_m) \longrightarrow A^* \in (G_m)$. Not every operator in (G_m) has SVEP. (Recall that hyponormal operators are (G_1): if every (G_m) operator had SVEP, then it would follow that both the forward unilateral shift and the backward unilateral shift have SVEP, which is known to be false.) A subclass of the class of (G_m) operators, the so called class of locally $-$ (G_m) operators (defined as below) acting on a reflexive Banach, has SVEP [12, Proposition 2]. For $A \in B(\mathcal{H})$ and an arbitrary closed subset F of the set $\mathbf{C}$ of complex numbers, let $X_A(F) = \{x \in \mathcal{X} : (A-\lambda)f_x(\lambda) \equiv x$ for some analytic function $f_x : \mathbf{C} \setminus F \longrightarrow \mathcal{X}\}$. ($X_A(F)$ is an invariant linear manifold for A [1, p. 60].) Let m be a positive integer. We say that $A \in$ locally $-(G_m)$ (or, A satisfies a local growth condition of order m) if for every closed set $F \subset \mathbf{C}$ and every $x \in X_A(F)$ there exists an analytic function $f : \mathbf{C} \setminus F \longrightarrow \mathcal{X}$ such that $(A-\lambda)f(\lambda) \equiv x$ and $||f(\lambda)|| \leq K[\mathrm{dist}(\lambda, F)]^{-m}||x||$ for some $K > 0$ (independent of F and x). All hyponormal operators belong to locally$-(G_1)$ and spectral operators of type $m-1$ belong to locally $-(G_m)$. Evidently, locally $-(G_m) \Longrightarrow (G_m)$. The following argument shows that $H_0(A-\lambda) = (A-\lambda)^{-m}(0)$ for operators $A \in (G_m)$. Let $\lambda_0 \in \mathrm{iso}\sigma(A)$, and let $\Gamma = \{\lambda : |\lambda - \lambda_0| < \epsilon\} \subset \rho(A)$ for some $0 < \epsilon \leq \mathrm{dis}(\lambda_0, \sigma(A) \setminus \{\lambda_0\})$. Then

$$(A-\lambda_0)^m = \frac{1}{2\pi i}\int_\Gamma (\lambda_0 - \lambda)^m (\lambda - A)^{-1} d\lambda,$$

and

$$||(A-\lambda_0)^m|| \leq \frac{1}{2\pi}\int_\Gamma |\lambda_0 - \lambda|^m ||(\lambda - A)^{-1}|||d\lambda| \leq \frac{1}{2\pi}\epsilon^m \frac{K}{\epsilon^m} 2\pi\epsilon,$$

which tends to zero with ϵ.

Corollary 4.2 *If the Banach spaces $\mathcal{X}_i$, $1 \leq i \leq n$, are separable (resp., reflexive) and the upper triangular operator $A = (A_{ij})_{1 \leq i,j \leq n} \in B(\mathcal{X}^n)$ is such that each A_{ii} is paranormal (resp., $\in$ locally $-(G_m)$), then (i) $f(A)$ and $f(A^*)$ satisfy Weyl's theorem for every $f \in H(A)$ and (ii) A^* satisfies a-Weyl's theorem .*

Proof. $A \in \mathcal{T}(\mathcal{X}^n)$, A_{ii} has SVEP for all $1 \leq i \leq n$, $H_0(A_{ii}-\lambda) = (A_{ii}-\lambda)^{-1}$ if A_{ii} is paranormal and $H_0(A_{ii}-\lambda) = (A_{ii}-\lambda)^{-m}$ if $A_{ii} \in (G_m)$ for every $\lambda \in \mathrm{iso}\sigma(A_{ii})$. □

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