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A STUDY OF SOME ASPECTS OF TOPOLOGICAL GROUPS

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Abstract

The aim of this paper is to find a generalization of topological groups. The concept arises out of the investigation to obtain a group structure on the set $[X, Y]$, of homotopy classes of maps from a space X to a given space Y for all X which is natural with respect to X. We also study the generalized topological groups. Finally, associated with each generalized topological group we construct a contravariant functor from the homotopy category of pointed topological spaces and base point preserving continuous maps to the category of groups and homomorphisms.

1 Introduction

We recall that a topological group is a group G whose underlying set is equipped with a topology such that

(i) the multiplication $\mu : G \times G \to G$, $(x, y) \to xy$, is continuous if $G \times G$ has the product topology,

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(ii) the inversion map $\phi : G \to G$, $x \to x^{-1}$ is continuous.

It is known that if X is a topological group and $x_0 \in X$, then the fundamental group $\pi_1(X, x_0)$ is abelian. Consequently, if X is a space such that $\pi_1(X, x_0)$ is not belian, there is no way to make X a topological group.

Now the question is:

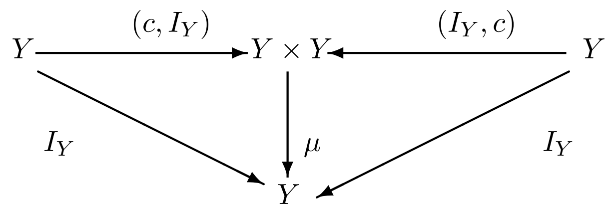
Does there exist a multiplication on [X,Y] making it a group for all Y? The answer is affirmative if Y is a topological group. If f, g :$X \to Y$ are continuous maps, define a product $f.g$ by the rule $(f.g)(x) = f(x)g(x)$, where the right hand side multiplication is the multiplication in the topological group. This law of composition is carried over to give an operation on the homotopy classes by the rule $[f] \circ [g] = [f.g]$ which admits $[X, Y]$ a group structure. On the other hand, if $X = S^1$, then the usual group structure on $[X, Y] = [S^1, Y] = \pi_1(Y)$ is a natural group operation.

We show that there exist some spaces Y which are not topological groups but $[X, Y]$ admits group structure for all X. This introduces the concept of generalized topological groups generalizing the concept of topological groups.

2 Preliminaries and Definitions

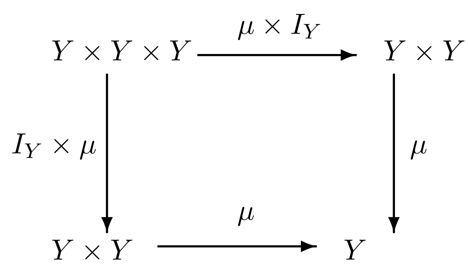
The concept of an H-space arose as a generalization of that of a topological group. The essential feature which is retained is a continuous multiplication with a unit. There is a significant class of spaces which are H-spaces but not topological groups. Some of the techniques which apply to topological groups can be applied to H-spaces, but not all. From the point of view of homotopy theory, it is not the existence of a continuous inverse which is the important distinguishing feature, but rather the associativity of multiplication. If we consider S^0, S^1, S^3 and S^7 as the real, complex, quaternionic and Cayley numbers of unit norm, these spaces have continuous multiplication. The multiplication in the first three cases are associative but not associative in the last case.

Definition 2.1 *An H-space is a pointed topological space* (Y, y_0) *with a continuous multiplication* $\mu : Y \times Y \to Y$ *such that the constant map* $c : Y \to y_0 \in Y$ *is a homotopy identity i.e. the diagram*



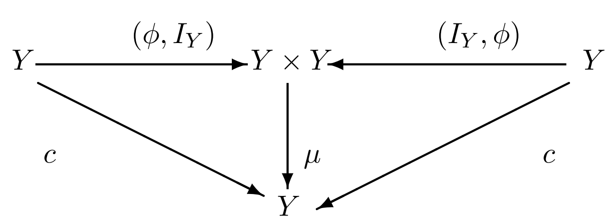
is homotopic commutative.
i.e. $\mu \circ (c, I_Y) \simeq I_Y \simeq \mu \circ (I_Y, c)$
i.e. $[\mu \circ (c, I_Y)] = I_Y = [\mu \circ (I_Y, c)]$*. Thus* (Y, y_0) *is an H-space if there is a continuous multiplication* $\mu : Y \times Y \to Y$ *such that* $\mu \circ j_1 \simeq I_Y \simeq \mu \circ j_2$ *where* $j_1, j_2 : Y \to Y \times Y$ *are injections defined by* $j_1(y) = (y, y_0)$ *and* $j_2(y) = (y_0, y)$*.*

Definition 2.2 *An H-space with multiplication* μ *is said to be homotopic associative if the following diagram*



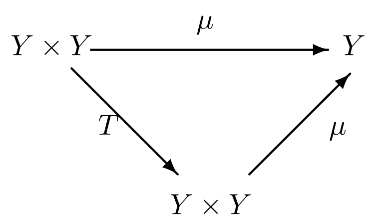
is homotopic commutative
i.e. $\mu \circ (\mu \times I_Y) \simeq \mu \circ (I_Y \times \mu)$
i.e. $[\mu \circ (\mu \times I_Y)] = [\mu \circ (I_Y \times \mu)]$*.*

Definition 2.3 *An H-space* Y *with multiplication* μ *is said to have an inverse* $\phi : (Y, y_0) \to (Y, y_0)$ *if the following diagram*



is homotopic commutative, where $c : Y \to y_0 \in Y$ *is the constant map.*
i.e. $\mu \circ (\phi, I_Y) \simeq c \simeq \mu \circ (I_Y, \phi)$
i.e. $[\mu \circ (\phi, I_Y)] = [c] = [\mu \circ (I_Y, \phi)]$.

Definition 2.4 *An H-space* Y *with multiplication* μ *is said to be homotopy commutative if the following diagram*



is homotopic commutative.
i.e. $\mu \circ T \simeq \mu$, *where* T *is defined by* $T(x, y) = (y, x)$
i.e.$[\mu \circ T] = [\mu]$. *All the maps and homotopies are relative to the base points.*

Definition 2.5 *A pointed space* Y *with a continuous multiplication* $\mu : Y \times Y \to Y$ *is called a generalized topological monoid if* Y *is an associative H-space.It is sometimes written as an ordered pair* (Y, μ).

Definition 2.6 *A generalized topological monoid* (Y, μ) *is said to be a generalized topological group if there exists a homotopy inverse* $\phi : Y \to Y$.

We know that if $Y \approx Y'$, then Y is a topological group iff $Y'$is so. We now extend this result to a larger class of spaces.

Theorem 2.7 *Let* Y *and* Y' *be two pointed topological spaces such that* $Y \simeq Y'$. *Then* Y *is a generalized topological group iff* Y' *is also so. In other words generalized topological groups are homotopy type invariant in the sense that any space homotopy equivalent to a generalized topological group is also a generalized topological group.*

Proof : Let $Y \simeq Y'$. Then there exists continuous maps $f : Y \to Y'$ and $g : Y' \to Y$ such that $g \circ f \simeq I_Y$ and $f \circ g \simeq I_{Y'}$. Suppose Y is a generalized topological group with continuous multiplication μ and homotopy inverse ϕ. Define $\mu' : Y' \times Y' \to Y'$ as the composite

$$Y' \times Y' \xrightarrow{g \times g} Y \times Y \xrightarrow{\mu} Y \xrightarrow{f} Y'$$

i.e. $\mu' = f \circ \mu \circ (g \times g)$. This implies μ' is a continuous multiplication in Y'.
The composite

$$Y' \xrightarrow{(I_{Y'}, c')} Y' \times Y' \xrightarrow{\mu'} Y'$$

where $c' : Y' \to y_0 \in Y'$ is the same as a the composite

$$Y' \xrightarrow{g} Y \xrightarrow{(I_Y, c)} Y \times Y \xrightarrow{\mu} Y \xrightarrow{f} Y'$$

because $\mu' \circ (I_{Y'}, c') = f \circ \mu \circ (g \times g) \circ (I_{Y'}, c')$.
Now $(f \circ \mu) \circ (g \times g) \circ (I_{Y'}, c')(y') = (f \circ \mu) \circ (g \times g)(y', y_0')$.
$=(f \circ \mu) \circ (g(y'), y_0)) = (f \circ \mu) \circ (I_Y, c) \circ g(y')$
$\Rightarrow (f \circ \mu) \circ (g \times g) \circ (I_{Y'}, c') = (f \circ \mu) \circ (I_Y, c) \circ g$, $\forall y' \in Y'$
$\Rightarrow \mu' \circ (I_{Y'}, c') = (f \circ \mu) \circ (I_Y, c) \circ g$
$\Rightarrow \mu' \circ (I_{Y'}, c') \simeq f \circ g$, since $\mu \circ (I_Y, c) \simeq I_Y$
$\Rightarrow \mu' \circ (I_{Y'}, c') \simeq I_{Y'}$.
Similarly, $\mu' \circ (c', I_{Y'}) \simeq I_{Y'}$. We now show that,
(i)μ is a homotopy associative $\Rightarrow \mu'$ is also so.
(ii) ϕ homotopy inverse for Y $\Rightarrow \phi'$ is homotopy inverse for Y'
(i) To prove $\mu' = f \circ \mu \circ (g \times g)$ is homotopy equivalence, ie to prove $\mu' \circ (\mu' \times I_{Y'}) \simeq \mu' \circ (I_{Y'} \times \mu')$ ie to prove $[\mu' \circ (\mu' \times I_{Y'})] = [\mu' \circ (I_{Y'} \times \mu')]$
Let $y' \in Y'$, then $(\mu' \times I_{Y'})((y', y'), y') = (y'y', y')$ and $\mu' \circ (\mu' \times I_{Y'}) = \mu'(y'y', y') = y'y'y'$.
Similarly $(I_{Y'}, \mu')(y', (y', y')) = (y', y'y')$ and $\mu' \circ (I_{Y'}, \mu')((y', y'), y') = \mu'(y', y'y') = y'y'y'$.
Hence $\mu' \circ (\mu' \times I_{Y'}) \simeq \mu' \circ (I_{Y'}, \mu')$ ie μ' is homotopy associative.
(ii) Again we show that $\phi : Y \to Y$ is a homotopy inverse for Y
$\Rightarrow \phi' = f \circ \phi \circ g : Y' \to Y'$ is a homotopy inverse for Y'.
i.e. $\mu' \circ (\phi', I_{Y'}) \simeq c' \simeq \mu' \circ (I_{Y'}, \phi')$
Now, $\mu' \circ (\phi', I_{Y'})$
$= (f \circ \mu) \circ (g \times g) \circ (\phi', I_{Y'})$
$= (f \circ \mu) \circ (g \times g) \circ (f \circ \phi \circ g, I_{Y'})$

$= f \circ \mu \circ (g \circ f \circ \phi \circ g, g \circ I_{Y'})$
$= f \circ \mu \circ (I_Y \circ \phi \circ g, g)$
$= f \circ \mu \circ (\phi \circ g, g)$
$= f \circ \mu \circ (\phi, I_Y)g$
$\simeq f \circ c \circ g$
$\simeq c'$, because

$$Y' \xrightarrow{\;g\;} Y \xrightarrow{\;c\;} Y \xrightarrow{\;f\;} Y'$$

where $y' \in Y'$ implies $(f \circ c \circ g)(y') = (f \circ c)(g(y')) = (f \circ c)(y) = f(c(y)) = f(y_0) = y'_0 = c'(y')$, $\forall y' \in Y'$.
Similarly $\mu' \circ (I_{Y'}, \phi') \simeq c'$
$\Rightarrow \phi'$ is homotopy inverse for Y'. Consequently (Y, μ) is a generalized topological group.
$\Rightarrow (Y', \mu')$ is also a generalized topological group.
Similarly (Y', μ') is a generalized topological group $\Rightarrow (Y, \mu)$ is also so.

Definition 2.8 *A continuous map $f : Y \to Y'$ between two generalized topological groups (Y, μ) and (Y', μ') is said to be a homotopy homomorphism if the following diagram is homotopy commutative.*

$$\begin{array}{ccc} Y \times Y & \xrightarrow{\;\mu\;} & Y \\ {\scriptstyle f \times f}\downarrow & & \downarrow{\scriptstyle f} \\ Y' \times Y' & \xrightarrow{\;\mu'\;} & Y' \end{array}$$

i.e. $f \circ \mu \simeq \mu' \circ (f \times f)$.

Corollary 2.9 *The homotopy equivalence f and its homotopy inverse g (in Theorem 2.7) are both homotopy homomorphisms.*

Corollary 2.10 *If $Y \simeq Y'$, then Y' is a generalized topological monoid iff Y is also so.*

3 Characterization of generalized topological groups

In certain situation a generalized topological monoid Y has an inverse. In this section we identify some such situations.

Theorem 3.1 *Let Y be a generalized topological monoid with the continuous multiplication μ. If Y is a generalized topological group, then the map $\psi : Y \times Y \to Y \times Y$ defined by $\psi(x, y) = (x, xy)$ is a homotopy equivalence. If μ is homotopic commutative, its converse is also true.*

Proof: Let (Y, μ) be a generalized topological group with homotopy inverse ϕ. Consider the map $\rho : Y \times Y \to Y \times Y$ defined by $\rho(x, y) = (x, \phi(x)y)$. Now $(\psi \circ \rho)(x, y) = \psi(\rho(x, y)) = \psi(x, \phi(x)y) = (x, x\phi(x)y) \Rightarrow \psi \circ \rho \simeq Id$, since $\phi : Y \to Y$ is the homotopy inverse. Again $(\rho \circ \psi)(x, y) = \rho(x, xy) = (x, \phi(x)xy) \Rightarrow \rho \circ \psi \simeq Id$. Hence it follows that (Y, μ) is a generalized topological group $\Rightarrow \psi$ is a homotopy equivalence with homotopy inverse ρ. Conversely, let ψ be a homotopy equivalence with homotopy inverse ρ. Then $\psi \circ \rho \simeq Id \simeq \rho \circ \psi$.
Define $\phi : Y \to Y$ by $\phi = p_2 \circ \rho \circ j_1$ where $j_1 : Y \to Y \times Y$ is the inclusion map defined by $j_1(y) = (y, y_0)$ and $p_i : Y \times Y \to Y$ is the projection of $Y \times Y$ onto the first and the second factor respectively.
Now $p_1 \circ \psi(x, y) = p_1(x, xy) = x = p_1(x, y), \forall (x, y) \in Y \times Y$
$\Rightarrow p_1 \circ \psi = p_1$.
Again $(p_2 \circ \psi)(x, y) = p_2(x, xy) = xy = \mu(x, y), \forall (x, y) \in Y \times Y$
$\Rightarrow p_2 \circ \psi = \mu$.

Consequently $p_1 \simeq p_1 \Rightarrow p_1 \simeq p_1 \circ \psi \circ \rho$, since $\psi \circ \rho \simeq Id$ and $p_2 \simeq p_2 \circ Id \simeq p_2 \circ \psi \circ \rho = \mu \circ \rho$ and $p_1 \circ \rho \circ j_1 = p_1 \circ \psi \circ \rho \circ j_1 \simeq p_1 \circ j_1 = Id$. Hence $\mu \circ (Id, \phi) = \mu \circ (p_1 \circ \rho \circ j_1, p_2 \circ \rho \circ j_1) = \mu \circ (p_1, p_2) \circ (\rho \circ j_1) = \mu \circ \rho \circ j_1 \simeq p_2 \circ j_1 = c$
$\Rightarrow \phi$ is the right homotopy inverse. Similarly, $\mu \circ (\phi, Id) = \mu \circ (p_2 \circ \rho \circ j_1, p_1 \circ \rho \circ j_1) = \mu \circ (p_2, p_1) \circ (\rho \circ j_1) = \mu \circ T \circ (\rho \circ j_1) \simeq \mu \circ \rho \circ j_1 = c$
$\Rightarrow \phi$ is the left homotopy inverse.
Hence $(Y, \mu)$is a generalized topological monoid such that $\phi : Y \to Y$ is both sided homotopy inverse. Consequently, (Y, μ) is a generalized topological group.

Corollary 3.2 *Let Y be a generalized topological monoid with homotopy commutative continuous multiplication μ.Then Y is a generalized topological group iff $\psi : Y \times Y \to Y \times Y$, defined by $\psi(x, y) = (x, xy)$ is a homotopy equivalence.*

Corollary 3.3 *Let Y be a generalized topological monoid. Then Y is a generalized topological group iff both of the two maps $\psi_1, \psi_2 : Y\times Y \to Y\times Y$, defined by $\psi_1(x, y) = (xy, x)$ and $\psi_2(x, y) = (xy, y)$, for all $x, y \in Y$ are homotopy equivalence of $Y \times Y$ into itself.*

Theorem 3.4 *Let Y be a CW-complex and the weak topology of the product complex $Y \times Y$ be the ordinary product topology. Then Y is a generalized topological group iff Y is generalized topological monoid.*

Proof: If Y is a generalized topological group, then it is automatically a generalized topological monoid. For its converse, let (Y, μ) be a generalized topological monoid. Consider the map $\psi_1 : Y \times Y \to Y \times Y$ defined by $\psi_1(x, y) = (xy, x)$. Then ψ_1 induces homomorphisms $\psi_1^* : \pi_n(Y \times Y) \to \pi_n(Y \times Y)$ for all positive integer n. We claim that ψ_1^* is an isomorphism.
Let $p_i : Y\times Y \to Y$ is the projection of $Y\times Y$ onto the first and the second factor respectively. Then p_i induces $p_i^* : \pi_n(Y \times Y) \to \pi_n(Y)$. Let j_1, j_2 be the natural inclusions from $Y \to Y \times Y$ defined by $j_1(y) = (y, y_0)$ and $j_2(y) = (y_0, y)$. Then j_1, j_2 induces $j_1^*, j_2^* : \pi_n(Y) \to \pi_n(Y \times Y)$. Then we have the following two isomorphisms between the groups $\pi_n(Y \times Y)$ and $\pi_n(Y) \oplus \pi_n(Y)$(the direct sum of two groups):
$(p_1^*, p_2^*) : \pi_n(Y \times Y) \cong \pi_n(Y) \oplus \pi_n(Y)$ and
$(j_1^*, j_2^*) : \pi_n(Y) \oplus \pi_n(Y) \cong \pi_n(Y \times Y)$
Now from the definition of ψ_1 it follows that
$(p_1^*, p_2^*) \circ \psi_1^* \circ j_1^*(\gamma) = (p_1^*, p_2^*) \circ \psi_1^*(\gamma, 0) = (p_1^*, p_2^*)(\gamma+0, \gamma) = (p_1^*, p_2^*)(\gamma, \gamma) = (\gamma, \gamma)$ and
$(p_1^*, p_2^*) \circ \psi_1^* \circ j_2^*(\delta) = (p_1^*, p_2^*) \circ \psi_1^*(\delta, 0) = (p_1^*, p_2^*)(\delta + 0, 0) = (p_1^*, p_2^*)(\delta, 0) = (\delta, 0)$ for all $\gamma, \delta \in \pi_n(Y)$
Hence $(p_1^*, p_2^*)\circ\psi_1^*\circ(j_1^*\oplus j_2^*)(\gamma, \delta) = ((p_1^*, p_2^*)\circ\psi_1^*\circ j_1^*(\gamma), (p_1^*, p_2^*)\circ\psi_1^*\circ j_2^*(\delta)) = ((\gamma, \gamma), (\delta, 0)) = (\gamma + \delta, \gamma)$
Moreover $(p_1^*, p_2^*) \circ \psi_1^* \circ (j_1^* \oplus j_2^*)(\gamma, \delta) = (0, 0)$, the zero element of $\pi_n(Y) \oplus \pi_n(Y)$, then $(\gamma + \delta, \gamma) = (0, 0)$ implies $\gamma = 0$, $\delta = 0$.
Hence it follows that ψ_1^* is an isomorphism of $\pi_n(Y\times Y)$ onto itself and it follows from a theorem of J.H.C.Whitehead that ψ_1 is a homotopy equivalence, since $Y\times Y$ is a CW-complex by hypothesis. Similarly, $\psi_2 : Y\times Y \to Y\times Y$, defined by $(x, y) \to (xy, y)$ is a homotopy equivalence. Hence by corollary 3.3 to the theorem 3.1, it follows that Y is a generalized topological group.

Theorem 3.5 *Let Y be a generalized topological monoid. Then the set $[X,Y]$ admits a monoid structure natural with respect to X and admits a group structure with respect to X if Y is a generalized topological group.*

Proof: Let (Y, y_0) be a generalized topological monoid with multiplication μ. We show that for every (X, x_0) the set $[X, x_0; Y, y_0]$ admits the monoid structure under the product $[f].[g]$ be the homotopy class of the composite

$$X \xrightarrow{\triangle} X \times X \xrightarrow{f \times g} Y \times Y \xrightarrow{\mu} Y$$

i.e. $[f].[g] = [\mu \circ (f \times g) \circ \triangle]$ where $\triangle(x) = (x, x), \forall x \in X$. We claim that $[f].[g]$ is well defined.
Suppose $H : X \times I \to Y$ is a homotopy between f and f'
and $G : X \times I \to Y$ is a homotopy between g and g'. Define a homotopy $K : X \times I \to Y$ by $k_t(x) = K(x, t) = \mu(H(x, t), G(x, t))$ then $K_0 = K(x, 0) = \mu(H(x, 0), G(x, 0)) = \mu(f(x), g(x)) = \mu \circ (f \times g)(x, x) = \mu \circ (f \times g) \circ \triangle(x) \Rightarrow K_0 = \mu \circ (f \times g) \circ \triangle$. Similarly $K_1 = \mu \circ (f' \times g') \circ \triangle$. Consequently, $[f].[g] = [\mu \circ (f \times g) \circ \triangle]$ is well defined. μ is clearly associative and the homotopy class of the constant map $c : X \to y_0$ is the identity element . Consequently, $[X, Y]$ is a monoid.
Let $\alpha : X \to X'$ is a continuous map. Then $\alpha^* : [X', Y] \to [X, Y]$ defined by $\alpha^*[f'] = [f' \circ \alpha]$ is a homomorphism of monoids.
It is natural to ask when the monoid [X,Y] admits a group structure.
Let $\phi : Y \to Y$ be the homotopy inverse for μ and Y with respect to the identity [c]. Then the inverse of $[f] \in [X, Y]$ is given by $[f]^{-1} = [\phi \circ f]$. Then [X,Y] is a group natural with respect to X.

4 Functors associated with generalized topological groups

In this section we construct a functor $\prod^G$ associated with each generalized topological group G from the homotopy category of pointed topological spaces and base point preserving continuous maps Htp to the category of groups and homomorphisms Grp.

Theorem 4.1 *Let G be a generalized topological group with base point e, continuous multiplication $\mu : G \times G \to G$ and homotopy inverse $\phi : G \to G$. Then their exist a contravariant functor $\prod^G : Htp \to Grp$.*

Proof: Define the product $[f] \circ [g]$ on $[X, G]$ to be the homotopy class of the composites

$$X \xrightarrow{\triangle} X \times X \xrightarrow{f \times g} G \times G \xrightarrow{\mu} G$$

where $\triangle$ is the diagonal map given by $\triangle(x) = (x, x)$. Then $[X, G]$ admits a group structure with identity element is the homotopy class $[c]$ of the constant map $c : X \to e$ and the inverse of $[f]$ is given by $[f]^{-1} = [\phi \circ f]$. We define the object function by $\prod^G(X) = [X, G]$, which is a group.For a base point preserving continuous map $f : X \to Y$, we define $\prod^G(f) : \prod^G(Y) \to \prod^G(X)$ by $\prod^G(f)([\alpha]) = [\alpha \circ f]$, $\forall[\alpha] \in \prod^G(Y) = [Y, G]$. This gives the morphism function.Hence $\prod^G : Htp \to Grp$ is a contravariant functor.

Corollary 4.2 *For each homotopy associative topological monoid* $M, \prod^M$ *is a contravariant functor from Htp to the category of monoids and there homomorphisms.*

Proof: In this case, $[X, G]$ admits a monoid structure and hence the corollary follows.

Corollary 4.3 $\prod^G$ *is a homotopy type invariant.*

Proof: Let $f : X \to Y$ be a homotopy equivalence with homotopy inverse $g : Y \to X$. Then $f \circ g \simeq I_Y$ and $g \circ f \simeq I_X$ imply that $\prod^G(f \circ g)$ and $\prod^G(g \circ f)$ are both indentity automorphisms and hence $\prod^G(X)$ and $\prod^G(Y)$ are isomorphic groups.

Corollary 4.4 *Let G be a pointed topological space such that* $\prod^G$ *assumes values in Grp. Then G is a generalized topological group. Moreover, for any pointed space X, the group structures on* $\prod^G(X)$ *and [X,G] coincide.*

Corollary 4.5 *Let* $\alpha : G \to H$ *be a homomorphism of generalized topological groups. Then* α *induces a natural transformation* $N(\alpha) : \prod^G \to \prod^H$, *where* $N(\alpha)(X) : [X, G] \to [X, H]$ *is defined by* $N(\alpha)(X)([f]) = [\alpha \circ f], \forall[f] \in [X, G]$.

Corollary 4.6 *Let G be a CW-complex such that* $G \times G$ *is also a CW-complex. If G is a homotopy associative generalized topological group, then* $\prod^G : Htp \to Grp$ *is a contravariant functor.*

Proof: Under the given condition, G has an inverse and hence G is a generalized topological group. Consequently the corollary follows.

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