Faculty of Sciences and Mathematics
University of Niš

Available at:
www.pmf.ni.ac.yu/sajt/publikacije/publikacije_pocetna.html

Filomat **21:1** (2007), 113–128

ABOUT A CLASS OF METRICAL N-LINEAR CONNECTIONS ON THE 2-TANGENT BUNDLE

Gheorghe Atanasiu and Monica Purcaru

Abstract

In the paper herein we treat some problems concerning the metric structure on the 2-tangent bundle: T^2M. We determine the set of all metric semi-symmetric N-linear connections, in the case when the nonlinear connection N is fixed. We prove that the sets: $\mathcal{T}_N$ of the transformations of N-linear connection having the same nonlinear connections N and $\overset{ms}{\mathcal{T}}_N$ of the transformations of metric semi-symmetric N-linear connections, having the same nonlinear connection N, together with the composition of mappings are groups. We obtain some important invariants of the group $\overset{ms}{\mathcal{T}}_N$ and we give their properties. We also study the transformation laws of the torsion and curvature d-tensor fields, with respect to the transformations of the groups $\mathcal{T}_N$ and $\overset{ms}{\mathcal{T}}_N$.

[1]Received: August 17, 2005
[2]2000 *Mathematics Subject Classification.* 53C05, 53C15, 53C21, 53C60, 53B05, 53B35, 53B40.
[3]*Keywords and phrases.* 2-tangent bundle,N-linear connection, JN-linear connection, metric structure,metric N-linear connection, metric semi-symmetric N-linear connection, transformations group, invariants, torsion, curvature.

1 The N-and JN-linear connections on tangent bundle of order two

Let M be a real C^∞-manifold with n dimensions and (T^2M, π, M) its 2-tangent bundle, [1]. The local coordinates on 3n-dimensional manifold T^2M are denoted by $(x^i, y^{(1)i}, y^{(2)i}) = (x, y^{(1)}, y^{(2)}) = u$, $(i = 1, 2, ...n)$.

Let $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}})$ be the natural basis of the tangent space TT^2M at the point $u \in T^2M$ and let us consider the natural 2-tangent structure on T^2M, $J : \chi(T^2M) \to \chi(T^2M)$ given by:

(1.1) $J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^{(1)i}},\ J(\frac{\partial}{\partial y^{(1)i}}) = \frac{\partial}{\partial y^{(2)i}}, J(\frac{\partial}{\partial y^{(2)i}}) = 0.$

We denote with N a nonlinear connection on T^2M with the local coefficients $(\underset{1}{N}{}^i_j, \underset{2}{N}{}^i_j)$ $(i, j = 1, 2, ..., n)$, [7], [8].

Hence, the tangent space of T^2M in the point $u \in T^2M$ is given by the direct sum of the linear vector spaces:

(1.2) $T_uT^2M = N_0(u) \oplus N_1(u) \oplus V_2(u),\ \forall u \in T^2M.$

An adapted basis to the direct decomposition (1.2) is given by:

(1.3) $\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(2)i}}\},$

where:

$$(1.4)\quad \begin{aligned} &\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{1}{N}{}^j_i \frac{\partial}{\partial y^{(1)j}} - \underset{2}{N}{}^j_i \frac{\partial}{\partial y^{(2)j}}, \\ &\frac{\delta}{\delta y^{(1)i}} = \frac{\partial}{\partial y^{(1)i}} - \underset{1}{N}{}^j_i \frac{\partial}{\partial y^{(2)j}}, \quad \frac{\delta}{\delta y^{(2)i}} = \frac{\partial}{\partial y^{(2)i}}. \end{aligned}$$

Let us consider the dual basis of (1.3):

(1.5) $\{dx^i, \delta y^{(1)i}, \delta y^{(2)i}\},$

where

$$(1.6)\quad \begin{aligned} &\delta x^i = dx^i, \quad \delta y^{(1)i} = dy^{(1)i} + \underset{1}{N}{}^i_j dx^j, \\ &\delta y^{(2)i} = dy^{(2)i} + \underset{1}{N}{}^i_j dy^{(1)j} + (\underset{2}{N}{}^i_j + \underset{1}{N}{}^i_m \underset{1}{N}{}^m{}_j) dx^j. \end{aligned}$$

Definition 1.1. *([1]-[3]) A linear connection D on T^2M, $D : \chi(T^2M)\times \chi(T^2M) \to \chi(T^2M)$ is called an N-linear connection on T^2M if it preserves by parallelism the horizontal and vertical distributions N_0, N_1 and V_2 on T^2M.*

An N-linear connection D on T^2M is characterized by its coefficients in the adapted basis (1.3) in the form:

$$(1.7)\quad \begin{aligned}
&D_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^j} = \underset{(00)}{L}{}^i{}_{jk}\frac{\delta}{\delta x^i},\ D_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta y^{(1)j}} = \underset{(10)}{L}{}^i{}_{jk}\frac{\delta}{\delta y^{(1)i}},\\
&D_{\frac{\delta}{\delta x^k}}\frac{\partial}{\partial y^{(2)j}} = \underset{(20)}{L}{}^i{}_{jk}\frac{\partial}{\partial y^{(2)i}},\\
&D_{\frac{\delta}{\delta y^{(1)k}}}\frac{\delta}{\delta x^j} = \underset{(01)}{C}{}^i{}_{jk}\frac{\delta}{\delta x^i}, D_{\frac{\delta}{\delta y^{(1)k}}}\frac{\delta}{\delta y^{(1)j}} = \underset{(11)}{C}{}^i{}_{jk}\frac{\delta}{\delta y^{(1)i}},\\
&D_{\frac{\delta}{\delta y^{(1)k}}}\frac{\partial}{\partial y^{(2)j}} = \underset{(21)}{C}{}^i{}_{jk}\frac{\partial}{\partial y^{(2)i}},\\
&D_{\frac{\partial}{\partial y^{(2)k}}}\frac{\delta}{\delta x^j} = \underset{(02)}{C}{}^i{}_{jk}\frac{\delta}{\delta x^i}, D_{\frac{\partial}{\partial y^{(2)k}}}\frac{\delta}{\delta y^{(1)j}} = \underset{(12)}{C}{}^i{}_{jk}\frac{\delta}{\delta y^{(1)i}},\\
&D_{\frac{\partial}{\partial y^{(2)k}}}\frac{\partial}{\partial y^{(2)j}} = \underset{(22)}{C}{}^i{}_{jk}\frac{\partial}{\partial y^{(2)i}}.
\end{aligned}$$

The system of nine functions:

$$(1.8)\quad D\Gamma(N) = (\underset{(00)}{L}{}^i{}_{jk}, \underset{(10)}{L}{}^i{}_{jk}, \underset{(20)}{L}{}^i{}_{jk}, \underset{(01)}{C}{}^i{}_{jk}, \underset{(11)}{C}{}^i{}_{jk}, \underset{(21)}{C}{}^i{}_{jk}, \underset{(02)}{C}{}^i{}_{jk}, \underset{(12)}{C}{}^i{}_{jk}, \underset{(22)}{C}{}^i{}_{jk}),$$

are called the coefficients of the N-linear connection D.

Generally, an N-linear connection $D\Gamma(N)$ on T^2M is not compatible with the natural 2-tangent structure J given by (1.1).

Definition 1.2. *An N-linear connection D on T^2M is called JN-linear connection if it is absolut parallel with respect to D:*

$$(1.9)\quad D_X J = 0,\ \ \forall X \in \chi(T^2M).$$

Theorem 1.1. *(Gh.Atanasiu,[1]) A JN-linear connection on T^2M is characterized by the coefficints $JD\Gamma(N)$ given by (1.8),where:*

$$(1.10)\quad \begin{aligned}
&\underset{(00)}{L}{}^i{}_{jk} = \underset{(10)}{L}{}^i{}_{jk} = \underset{(20)}{L}{}^i{}_{jk}\ (= L^i{}_{jk}),\\
&\underset{(01)}{C}{}^i{}_{jk} = \underset{(11)}{C}{}^i{}_{jk} = \underset{(21)}{C}{}^i{}_{jk}\ (= \underset{(1)}{C}{}^i{}_{jk}),\quad \underset{(02)}{C}{}^i{}_{jk} = \underset{(12)}{C}{}^i{}_{jk} = \underset{(22)}{C}{}^i{}_{jk}\ (= \underset{(2)}{C}{}^i{}_{jk}).
\end{aligned}$$

It results that a $JD\Gamma(N)$- linear connection on T^2M has three essentially coefficients:

$$(1.11)\quad JD\Gamma(N) = (L^i{}_{jk}, \underset{(1)}{C}{}^i{}_{jk}, \underset{(2)}{C}{}^i{}_{jk}).$$

Obvious, the geometrical theory on 2- tangent bundle (T^2M, π, M) with the N- linear connection [1]-[3], [15], generalize that with the JN-linear connection (cf.with R.Miron and Gh.Atanasiu [5]-[8]; see, also M.Purcaru [12],

[13]).

In the following we use the N-linear connections, only.

2 The transformations of the d-tensors of torsion and curvature in $\mathcal{T}_N$

Let $\mathcal{T}_N$ be the set of transformations of N-linear connections, corresponding to the same nonlinear connection N:

$$\mathcal{T}_N = \{t(0,0,\underset{(00)}{B}{}^i{}_{jk},\underset{(10)}{B}{}^i{}_{jk},\underset{(20)}{B}{}^i{}_{jk},\underset{(01)}{D}{}^i{}_{jk},\underset{(11)}{D}{}^i{}_{jk},\underset{(21)}{D}{}^i{}_{jk},\underset{(02)}{D}{}^i{}_{jk},\underset{(12)}{D}{}^i{}_{jk},\underset{(22)}{D}{}^i{}_{jk}) \in \mathcal{T}\},$$

where $\mathcal{T}$ is the set of the transformations of N-linear connections to $\bar{N}$-linear connections.

We have:

Theorem 2.1. *The set $\mathcal{T}_N$ of the transformations of N-linear connections to N-linear connections, together with the composition of mappings is an Abelian group. This group, $\mathcal{T}_N$, acts effectively and transitivelly on the set of N-linear connections.*

Proof. Let $t(0,0,\underset{(00)}{B}{}^i{}_{jk},\underset{(10)}{B}{}^i{}_{jk},\underset{(20)}{B}{}^i{}_{jk},\underset{(01)}{D}{}^i{}_{jk},\underset{(11)}{D}{}^i{}_{jk},\underset{(21)}{D}{}^i{}_{jk},\underset{(02)}{D}{}^i{}_{jk},\underset{(12)}{D}{}^i{}_{jk},\underset{(22)}{D}{}^i{}_{jk}) : D\Gamma(N) \to D\bar{\Gamma}(N)$ be a transformation from $\mathcal{T}_N$ given by (2.1):

$$\begin{aligned} &\underset{\beta}{\bar{N}}{}^i{}_j = \underset{\beta}{N}{}^i{}_j, \quad \underset{(\alpha 0)}{\bar{L}}{}^i{}_{jk} = \underset{(\alpha 0)}{L}{}^i{}_{jk} - \underset{(\alpha 0)}{B}{}^i{}_{jk},\\ &\underset{(\alpha 1)}{\bar{C}}{}^i{}_{jk} = \underset{(\alpha 1)}{C}{}^i{}_{jk} - \underset{(\alpha 1)}{D}{}^i{}_{jk}, \quad \underset{(\alpha 2)}{\bar{C}}{}^i{}_{jk} = \underset{(\alpha 2)}{C}{}^i{}_{jk} - \underset{(\alpha 2)}{D}{}^i{}_{jk},\\ &(\alpha = 0,1,2;\ \beta = 1,2). \end{aligned} \tag{2.1}$$

The composition of two transformations from $\mathcal{T}_N$ is a transformation from $\mathcal{T}_N$, given by: $t(0,0,\underset{(00)}{\bar{B}}{}^i{}_{jk},\underset{(10)}{\bar{B}}{}^i{}_{jk},\underset{(20)}{\bar{B}}{}^i{}_{jk},\underset{(01)}{\bar{D}}{}^i{}_{jk},\underset{(11)}{\bar{D}}{}^i{}_{jk},\underset{(21)}{\bar{D}}{}^i{}_{jk},\underset{(02)}{\bar{D}}{}^i{}_{jk},\underset{(12)}{\bar{D}}{}^i{}_{jk},\underset{(22)}{\bar{D}}{}^i{}_{jk}) \circ t(0,0,\underset{(00)}{B}{}^i{}_{jk},\underset{(10)}{B}{}^i{}_{jk},\underset{(20)}{B}{}^i{}_{jk},\underset{(01)}{D}{}^i{}_{jk},\underset{(11)}{D}{}^i{}_{jk},\underset{(21)}{D}{}^i{}_{jk},\underset{(02)}{D}{}^i{}_{jk},\underset{(12)}{D}{}^i{}_{jk},\underset{(22)}{D}{}^i{}_{jk}) = t(0,0,\underset{(00)}{B}{}^i{}_{jk} + \underset{(00)}{\bar{B}}{}^i{}_{jk},\underset{(10)}{B}{}^i{}_{jk} + \underset{(10)}{\bar{B}}{}^i{}_{jk},\underset{(20)}{B}{}^i{}_{jk} + \underset{(20)}{\bar{B}}{}^i{}_{jk},\underset{(01)}{D}{}^i{}_{jk} + \underset{(01)}{\bar{D}}{}^i{}_{jk},\underset{(11)}{D}{}^i{}_{jk} + \underset{(11)}{\bar{D}}{}^i{}_{jk},\underset{(21)}{D}{}^i{}_{jk} + \underset{(21)}{\bar{D}}{}^i{}_{jk},\underset{(02)}{D}{}^i{}_{jk} + \underset{(02)}{\bar{D}}{}^i{}_{jk},\underset{(12)}{D}{}^i{}_{jk} + \underset{(12)}{\bar{D}}{}^i{}_{jk},\underset{(22)}{D}{}^i{}_{jk} + \underset{(22)}{\bar{D}}{}^i{}_{jk})$.

The inverse of a transformation from $\mathcal{T}_N$ is the transformation:

$$t(0,0,-\underset{(00)}{B}{}^i{}_{jk},-\underset{(10)}{B}{}^i{}_{jk},-\underset{(20)}{B}{}^i{}_{jk},-\underset{(01)}{D}{}^i{}_{jk},-\underset{(11)}{D}{}^i{}_{jk},-\underset{(21)}{D}{}^i{}_{jk},-\underset{(02)}{D}{}^i{}_{jk},-\underset{(12)}{D}{}^i{}_{jk},-\underset{(22)}{D}{}^i{}_{jk})$$

$: D\Gamma(N) \to D\bar{\Gamma}(N)$.

The transformation (2.1) preserves all the N-linear connections D if $\underset{(\alpha 0)}{B}{}^{i}{}_{jk} = \underset{(\alpha 0)}{D}{}^{i}{}_{jk} = 0, (\alpha = 0, 1, 2)$. Therefore $\mathcal{T}_N$ acts effectively on the set of N-linear connections.

Firstly, we shall study the transformations of the d-tensors of torsion of $D\Gamma(N)$ (see, (7.2) and (7.5), [1]) by a transformation (2.1). We obtain:

Proposition 2.1. *The transformations of the Abelian group $\mathcal{T}_N$, given by (2.1) lead to the transformations of the d-tensors of torsion in the following way:*

$$
\begin{aligned}
&\underset{(0)}{\overset{\alpha}{\bar{T}}}{}^{i}{}_{jk} = \underset{(0)}{\overset{\alpha}{T}}{}^{i}{}_{jk} + (\underset{(\alpha 0)}{B}{}^{i}{}_{kj} - \underset{(\alpha 0)}{B}{}^{i}{}_{jk}), \quad \underset{(\beta)}{\overset{\alpha}{\bar{S}}}{}^{i}{}_{jk} = \underset{(\beta)}{\overset{\alpha}{S}}{}^{i}{}_{jk} + (\underset{(\alpha\beta)}{D}{}^{i}{}_{kj} - \underset{(\alpha\beta)}{D}{}^{i}{}_{jk}),\\
&\underset{(21)}{\bar{Q}}{}^{i}{}_{jk} = \underset{(21)}{Q}{}^{i}{}_{jk} - \underset{(12)}{D}{}^{i}{}_{jk}, \quad \underset{(22)}{\overset{\alpha}{\bar{Q}}}{}^{i}{}_{jk} = \underset{(22)}{\overset{\alpha}{Q}}{}^{i}{}_{jk} + \underset{(\alpha 1)}{D}{}^{i}{}_{kj}, \quad \underset{(12)}{\bar{S}}{}^{i}{}_{jk} = \underset{(12)}{S}{}^{i}{}_{jk},\\
&\underset{(0\beta)}{\bar{R}}{}^{i}{}_{jk} = \underset{(0\beta)}{R}{}^{i}{}_{jk}, \quad \underset{(\beta\beta)}{\overset{\alpha}{\bar{P}}}{}^{i}{}_{jk} = \underset{(\beta\beta)}{\overset{\alpha}{P}}{}^{i}{}_{jk} + \underset{(\alpha 0)}{B}{}^{i}{}_{kj}, \quad \underset{(\beta 0)}{\bar{P}}{}^{i}{}_{jk} = \underset{(\beta 0)}{P}{}^{i}{}_{jk} - \underset{(0\beta)}{D}{}^{i}{}_{jk},\\
&\underset{(12)}{\bar{P}}{}^{i}{}_{jk} = \underset{(12)}{P}{}^{i}{}_{jk}, \quad \underset{(21)}{\bar{P}}{}^{i}{}_{jk} = \underset{(21)}{P}{}^{i}{}_{jk}, \quad (\alpha = 0, 1, 2;\ \beta = 1, 2).
\end{aligned}
\tag{2.2}
$$

Now, we shall study the transformations of the d-tensors of curvature of $D\Gamma(N)$ (see, (7.11),[1]) by a transformation (2.1). We get:

Proposition 2.2. *The transformations of the Abelian group $\mathcal{T}_N$, given by (2.1) lead to the transformations of the d-tensors of curvature in the following way:*

$$
\begin{aligned}
\underset{(0\alpha)}{\bar{R}}{}_{h}{}^{i}{}_{jk} = {}& \underset{(0\alpha)}{R}{}_{h}{}^{i}{}_{jk} - \underset{(\alpha 1)}{D}{}^{i}{}_{hs} \underset{(01)}{R}{}^{s}{}_{jk} - \underset{(\alpha 2)}{D}{}^{i}{}_{hs} \underset{(02)}{R}{}^{s}{}_{jk} - \underset{(\alpha 0)}{B}{}^{i}{}_{hs} \underset{(0)}{\overset{\alpha}{T}}{}^{s}{}_{jk} +\\
&+\mathcal{A}_{jk}\{-\underset{(\alpha 0)}{B}{}^{i}{}_{hj|\alpha k} + \underset{(\alpha 0)}{B}{}^{s}{}_{hj} \underset{(\alpha 0)}{B}{}^{i}{}_{sk}\},
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
\underset{(1\alpha)}{\bar{P}}{}_{h}{}^{i}{}_{jk} = {}& \underset{(1\alpha)}{P}{}_{h}{}^{i}{}_{jk} - \underset{(\alpha 1)}{D}{}^{i}{}_{hs} \underset{(11)}{\overset{\alpha}{P}}{}^{s}{}_{jk} - \underset{(\alpha 2)}{D}{}^{i}{}_{hs} \underset{(12)}{P}{}^{s}{}_{jk} - \underset{(\alpha 0)}{B}{}^{i}{}_{hs} \underset{(\alpha 1)}{C}{}^{s}{}_{jk} +\\
&+ \underset{(\alpha 0)}{L}{}^{s}{}_{kj} \underset{(\alpha 1)}{D}{}^{i}{}_{hs} - \underset{(\alpha 0)}{B}{}^{i}{}_{hj} \overset{(1)}{|}{}_{\alpha k} + \underset{(\alpha 1)}{D}{}^{i}{}_{hk|\alpha j} + \underset{(\alpha 0)}{B}{}^{s}{}_{hj} \underset{(\alpha 1)}{D}{}^{i}{}_{sk} -\\
&- \underset{(\alpha 1)}{D}{}^{s}{}_{hk} \underset{(\alpha 0)}{B}{}^{i}{}_{sj} + \underset{(\alpha 1)}{C}{}^{i}{}_{hs} \underset{(\alpha 0)}{B}{}^{s}{}_{kj} - \underset{(\alpha 1)}{D}{}^{i}{}_{hs} \underset{(\alpha 0)}{B}{}^{s}{}_{kj},
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
\underset{(2\alpha)}{\bar{P}}{}_{h}{}^{i}{}_{jk} = {}& \underset{(2\alpha)}{P}{}_{h}{}^{i}{}_{jk} - \underset{(\alpha 1)}{D}{}^{i}{}_{hs} \underset{(21)}{P}{}^{s}{}_{jk} - \underset{(\alpha 2)}{D}{}^{i}{}_{hs} \underset{(22)}{\overset{\alpha}{P}}{}^{s}{}_{jk} - \underset{(\alpha 0)}{B}{}^{i}{}_{hs} \underset{(\alpha 2)}{C}{}^{s}{}_{jk} +\\
&+ \underset{(\alpha 0)}{L}{}^{s}{}_{kj} \underset{(\alpha 2)}{D}{}^{i}{}_{hs} - \underset{(\alpha 0)}{B}{}^{i}{}_{hj} \overset{(2)}{|}{}_{\alpha k} + \underset{(\alpha 2)}{D}{}^{i}{}_{hk|\alpha j} + \underset{(\alpha 0)}{B}{}^{s}{}_{hj} \underset{(\alpha 2)}{D}{}^{i}{}_{sk} -
\end{aligned}
\tag{2.5}
$$

$$- \underset{(\alpha 2)}{D^s{}_{hk}} \underset{(\alpha 0)}{B^i{}_{sj}} + \underset{(\alpha 2)}{C^i{}_{hs}} \underset{(\alpha 0)}{B^s{}_{kj}} - \underset{(\alpha 2)}{D^i{}_{hs}} \underset{(\alpha 0)}{B^s{}_{kj}},$$

$$(2.6) \quad \underset{(2\alpha)}{\bar{Q}_h{}^i{}_{jk}} = \underset{(2\alpha)}{Q_h{}^i{}_{jk}} - \underset{(\alpha 2)}{C^s{}_{jk}} \underset{(\alpha 1)}{D^i{}_{hs}} + \underset{(\alpha 1)}{C^s{}_{kj}} \underset{(\alpha 2)}{D^i{}_{hs}} - \underset{(\alpha 1)}{D^i{}_{hj}} \overset{(2)}{|}_{\alpha k} +$$

$$+ \underset{(\alpha 2)}{D^i{}_{hk}} \overset{(1)}{|}_{\alpha j} + \underset{(\alpha 1)}{D^s{}_{hj}} \underset{(\alpha 2)}{D^i{}_{sk}} - \underset{(\alpha 2)}{D^s{}_{hk}} \underset{(\alpha 1)}{D^i{}_{sj}} - \underset{(\alpha 2)}{D^i{}_{hs}} \underset{(21)}{P^s{}_{jk}},$$

$$(2.7) \quad \underset{(\beta\alpha)}{\bar{S}_h{}^i{}_{jk}} = \underset{(\beta\alpha)}{S_h{}^i{}_{jk}} - \underset{(\alpha\beta)}{D^i{}_{hs}} \underset{(\beta)}{\overset{\alpha}{S}{}^s{}_{jk}} + \mathcal{A}_{jk}\{- \underset{(\alpha\beta)}{D^i{}_{hj}} \overset{(\beta)}{|}_{\alpha k} +$$

$$+ \underset{(\alpha\beta)}{D^s{}_{hj}} \underset{(\alpha\beta)}{D^i{}_{sk}} \} - \underset{(\alpha 2)}{D^i{}_{hs}} \underset{(\beta 2)}{R^s{}_{jk}}, \ (\alpha = 0,1,2;\ \beta = 1,2),$$

where $\mathcal{A}_{ij}\{...\}$ denotes the alternate summation.

We shall consider the tensor fields:

$$(2.8) \quad \underset{(0\alpha)}{\mathbb{K}_h{}^i{}_{jk}} = \underset{(0\alpha)}{R_h{}^i{}_{jk}} - \underset{(\alpha 1)}{C^i{}_{hs}} \underset{(01)}{R^s{}_{jk}} - \underset{(\alpha 2)}{C^i{}_{hs}} \underset{(02)}{R^s{}_{jk}},$$

$$(2.9) \ \underset{(1\alpha)}{\mathbb{P}_h{}^i{}_{jk}} = \mathcal{A}_{jk}\{ \underset{(1\alpha)}{P_h{}^i{}_{jk}} - \underset{(\alpha 1)}{C^i{}_{hs}} \frac{\delta \underset{1}{N^s{}_j}}{\delta y^{(1)k}} - \underset{(\alpha 2)}{C^i{}_{hs}} (\underset{1}{N^s{}_m} \frac{\delta \underset{1}{N^m{}_j}}{\delta y^{(1)k}} +$$

$$+ \frac{\delta \underset{2}{N^s{}_j}}{\delta y^{(1)k}} - \frac{\delta \underset{2}{N^s{}_k}}{\delta y^{(1)j}})\},$$

$$(2.10) \ \underset{(2\alpha)}{\mathbb{P}_h{}^i{}_{jk}} = \mathcal{A}_{jk}\{ \underset{(2\alpha)}{P_h{}^i{}_{jk}} - \underset{(\alpha 1)}{C^i{}_{hs}} \frac{\partial \underset{1}{N^s{}_j}}{\delta y^{(2)k}} - \underset{(\alpha 2)}{C^i{}_{hs}} (\underset{1}{N^s{}_m} \frac{\partial \underset{1}{N^m{}_j}}{\delta y^{(2)k}} + \frac{\partial \underset{2}{N^s{}_j}}{\delta y^{(2)k}})\},$$

$$(2.11) \ \underset{(2\alpha)}{\mathbb{Q}_h{}^i{}_{jk}} = \mathcal{A}_{jk}\{ \underset{(2\alpha)}{Q_h{}^i{}_{jk}} + \underset{(\alpha 2)}{C^i{}_{hs}} \frac{\partial \underset{1}{N^s{}_j}}{\delta y^{(2)k}}\},$$

$$(2.12) \ \underset{(\beta\alpha)}{\mathbb{S}_h{}^i{}_{jk}} = \underset{(\beta\alpha)}{S_h{}^i{}_{jk}} - \underset{(\alpha 2)}{C^i{}_{hs}} \underset{(\beta 2)}{R^s{}_{jk}}, (\alpha = 0,1,2;\ \beta = 1,2).$$

Proposition 2.3. *By a transformation of the Abelian group $\mathcal{T}_N$, given by (2.1), the tensor fields: $\underset{(0\alpha)}{\mathbb{K}_h{}^i{}_{jk}}$, $\underset{(1\alpha)}{\mathbb{P}_h{}^i{}_{jk}}$, $\underset{(2\alpha)}{\mathbb{P}_h{}^i{}_{jk}}$, $\underset{(2\alpha)}{\mathbb{Q}_h{}^i{}_{jk}}$ and $\underset{(\beta\alpha)}{\mathbb{S}_h{}^i{}_{jk}}$, $(\alpha = 0,1,2;\ \beta = 1,2)$ are transformed according to the following laws:*

$$(2.13) \quad \underset{(0\alpha)}{\bar{\mathbb{K}}_h{}^i{}_{jk}} = \underset{(0\alpha)}{\mathbb{K}_h{}^i{}_{jk}} - \underset{(\alpha 0)}{B^i{}_{hs}} \underset{(0)}{\overset{\alpha}{T}{}^s{}_{jk}} + \mathcal{A}_{jk}\{- \underset{(\alpha 0)}{B^i{}_{hj|\alpha k}} + \underset{(\alpha 0)}{B^s{}_{kj}} \underset{(\alpha 0)}{B^i{}_{sk}} \},$$

$$(2.14) \quad \underset{(\beta\alpha)}{\bar{\mathbb{P}}_h{}^i{}_{jk}} = \underset{(\beta\alpha)}{\mathbb{P}_h{}^i{}_{jk}} - 2 \underset{(\alpha\beta)}{D^i{}_{hs}} \underset{(0)}{\overset{\alpha}{T}{}^s{}_{jk}} - \underset{(\alpha 0)}{B^i{}_{hs}} \underset{(\beta)}{\overset{\alpha}{S}{}^s{}_{jk}} + \mathcal{A}_{jk}\{- \underset{(\alpha 0)}{B^i{}_{hj}} \overset{(\beta)}{|}_{\alpha k}$$

$$- \underset{(\alpha\beta)}{D^i{}_{hj|\alpha k}} + \underset{(\alpha 0)}{B^s{}_{hj}} \underset{(\alpha\beta)}{D^i{}_{sk}} + \underset{(\alpha\beta)}{D^s{}_{hj}} \underset{(\alpha 0)}{B^i{}_{sk}} + \underset{(\alpha\beta)}{D^i{}_{hs}} \underset{(\alpha 0)}{B^s{}_{jk}} - \underset{(\alpha\beta)}{C^i{}_{hs}} \underset{(\alpha 0)}{B^s{}_{jk}} \},$$

$$(2.15)\ \underset{(2\alpha)}{\bar{\mathbb{Q}}}{}_h{}^i{}_{jk} = \underset{(2\alpha)}{\mathbb{Q}}{}_h{}^i{}_{jk} + {}^{\alpha} \underset{(2)}{S}{}^s{}_{jk} \underset{(\alpha 1)}{D}{}^i{}_{hs} - \underset{(1)}{\overset{\alpha}{S}}{}^s{}_{jk} \underset{(\alpha 2)}{D}{}^i{}_{hs} + \mathcal{A}_{jk}\{ \underset{(\alpha 1)}{D}{}^i{}_{hj} \overset{(2)}{|}_{\alpha k}$$

$$+ \underset{(\alpha 2)}{D}{}^i{}_{hk} \overset{(1)}{|}_{\alpha j} + \underset{(\alpha 1)}{D}{}^s{}_{hj} \underset{(\alpha 2)}{D}{}^i{}_{sk} - \underset{(\alpha 2)}{D}{}^s{}_{hk} \underset{(\alpha 1)}{D}{}^i{}_{sj} \},$$

$$(2.16)\ \underset{(\beta\alpha)}{\bar{\mathbb{S}}}{}_h{}^i{}_{jk} = \underset{(\beta\alpha)}{\mathbb{S}}{}_h{}^i{}_{jk} - \underset{(\alpha\beta)}{D}{}^i{}_{hs} \underset{(\beta)}{\overset{\alpha}{S}}{}^s{}_{jk} + \mathcal{A}_{jk}\{- \underset{(\alpha\beta)}{D}{}^i{}_{hj} \overset{(\beta)}{|}_{\alpha k} + \underset{(\alpha\beta)}{D}{}^s{}_{hj} \underset{(\alpha\beta)}{D}{}^i{}_{sk} \},$$

$(\alpha = 0, 1, 2;\ \beta = 1, 2)$.

3 Metric semi-symmetric N-linear connections on 2-tangent bundle

Definition 3.1. *([1]) A metric structure on the manifold T^2M is a symmetric covariant tensor field G of the type (0,2), which is non-degenerate at each point $u \in T^2M$ and of constant signature on T^2M.*

Let G be a metric structure on T^2M. Locally G looks as follows:

$$(3.1)\quad G = \underset{(0)}{g_{ij}}\, dx^i \otimes dx^j + \underset{(1)}{g_{ij}}\, \delta y^{(1)i} \otimes \delta y^{(1)j} + \underset{(2)}{g_{ij}}\, \delta y^{(2)i} \otimes \delta y^{(2)j}.$$

Definition 3.2. *([1]) An N-linear connection D on T^2M endowed with a metric structure G is said to be a metric N-linear connection if $D_X G = 0$ for every $X \in \mathcal{X}\left(T^2M\right)$.*

Proposition 3.1.*([1]) An N-linear connection D on T^2M endowed with a metric structure G is a metric N-linear connection if and only if:*

$$(3.2)\quad \begin{array}{lll} \underset{0}{D}{}^H_X G^H = 0, & \underset{0}{D}{}^{V_1}_X G^H = 0, & \underset{0}{D}{}^{V_2}_X G^H = 0, \\ \underset{\beta}{D}{}^H_X G^{V_\beta} = 0, & \underset{\beta}{D}{}^{V_1}_X G^{V_\beta} = 0, & \underset{\beta}{D}{}^{V_2}_X G^{V_\beta} = 0, \quad (\beta = 1, 2)\,. \end{array}$$

Translating the Proposition 3.1. in local coordinates we obtain:

Proposition 3.2.*([1]) An N-linear connection on T^2M is a metric N-linear connection if and only if:*

$$(3.3)\quad \underset{(\alpha)}{g}{}_{ij|\alpha k} = 0,\ \underset{(\alpha)}{g}{}_{ij} \overset{(1)}{|}_{\alpha k} = 0,\ \underset{(\alpha)}{g}{}_{ij} \overset{(2)}{|}_{\alpha k} = 0, (\alpha = 0, 1, 2)\,.$$

Theorem 3.1.*([1]) If the manifold T^2M is endowed with the metric* *structure G given by:*

$$(D_X G)(Y^H, Z^{V\beta}) = XG(Y^H, Z^{V\beta}) - G(D_X Y^H, Z^{V\beta}) - \\ -G(Y^H, D_X Z^{V\beta}) = 0, \tag{3.4}$$

then there exists on T^2M *a metric N-linear connection, depending only on G, whose h(hh)-,* $v_1(v_1v_1)$*- and* $v_2(v_2v_2)$*- tensors of torsion vanish. Its local coefficients* $D\overset{c}{\Gamma}(N) = (\underset{(\alpha 0)}{\overset{c}{L}}{}^i{}_{jk}, \underset{(\alpha 1)}{\overset{c}{C}}{}^i{}_{jk}, \underset{(\alpha 2)}{\overset{c}{C}}{}^i{}_{jk})$, $(\alpha = 0, 1, 2)$*, are as follows:*

$$\begin{aligned} \underset{(00)}{\overset{c}{L}}{}^i{}_{jk} &= \tfrac{1}{2} \underset{(0)}{g}{}^{il} (\delta_k \underset{(0)}{g}{}_{jl} + \delta_j \underset{(0)}{g}{}_{lk} - \delta_l \underset{(0)}{g}{}_{jk}), \\ \underset{(\beta 0)}{\overset{c}{L}}{}^i{}_{jk} &= \underset{(\beta\beta)}{B}{}^i{}_{kj} + \tfrac{1}{2} \underset{(\beta)}{g}{}^{il} (\delta_k \underset{(\beta)}{g}{}_{jl} - \underset{(\beta\beta)}{B}{}^m{}_{kj} \underset{(\beta)}{g}{}_{ml} - \underset{(\beta\beta)}{B}{}^m{}_{kl} \underset{(\beta)}{g}{}_{jm}), \\ \underset{(\delta 1)}{\overset{c}{C}}{}^i{}_{jk} &= \tfrac{1}{2} \underset{(\delta)}{g}{}^{il} \delta_{1k} \underset{(\delta)}{g}{}_{jl}, \quad \underset{(\varepsilon 2)}{\overset{c}{C}}{}^i{}_{jk} = \tfrac{1}{2} \underset{(\varepsilon)}{g}{}^{id} \dot{\partial}_{2k} \underset{(\varepsilon)}{g}{}_{jl}, \\ \underset{(\beta\beta)}{\overset{c}{C}}{}^i{}_{jk} &= \tfrac{1}{2} \underset{(\beta)}{g}{}^{il} (\delta_{\beta k} \underset{(\beta)}{g}{}_{jl} + \delta_{\beta j} \underset{(\beta)}{g}{}_{lk} - \delta_{\beta l} \underset{(\beta)}{g}{}_{jk}), \\ & (\delta = 0, 2;\ \beta = 1, 2;\ \varepsilon = 0, 1), \delta_{2i} = \dot{\partial}_{2i}. \end{aligned} \tag{3.5}$$

Definition 3.3. *The metric N-linear connection given by (3.5) is called the canonical N-linear connection associated with G.*

Let us associate to G the following operators of Obata type:

$$\underset{1}{\overset{\alpha}{\Omega}}{}^{ir}_{sj} = \tfrac{1}{2}(\delta^i_s \delta^r_j - \underset{(\alpha)}{g}{}_{sj} \underset{(\alpha)}{g}{}^{ir}), \quad \underset{2}{\overset{\alpha}{\Omega}}{}^{ir}_{sj} = \tfrac{1}{2}(\delta^i_s \delta^r_j + \underset{(\alpha)}{g}{}_{sj} \underset{(\alpha)}{g}{}^{ir}), \ (\alpha = 0, 1, 2). \tag{3.6}$$

There is inferred:

Proposition 3.3. *The Obata's operators have the following properties:*

$$\underset{1}{\overset{\alpha}{\Omega}}{}^{ir}_{sj} + \underset{2}{\overset{\alpha}{\Omega}}{}^{ir}_{sj} = \delta^i_s \delta^r_j, \tag{3.7}$$

$$\underset{1}{\overset{\alpha}{\Omega}}{}^{ir}_{sj} \underset{1}{\overset{\alpha}{\Omega}}{}^{sn}_{mr} = \underset{1}{\overset{\alpha}{\Omega}}{}^{in}_{mj}, \quad \underset{2}{\overset{\alpha}{\Omega}}{}^{ir}_{sj} \underset{2}{\overset{\alpha}{\Omega}}{}^{sn}_{mr} = \underset{2}{\overset{\alpha}{\Omega}}{}^{in}_{mj}, \quad \underset{1}{\overset{\alpha}{\Omega}}{}^{ir}_{sj} \underset{2}{\overset{\alpha}{\Omega}}{}^{sn}_{mr} = \underset{2}{\overset{\alpha}{\Omega}}{}^{ir}_{sj} \underset{1}{\overset{\alpha}{\Omega}}{}^{sn}_{mr} = 0, \tag{3.8}$$

$$\underset{1}{\overset{\alpha}{\Omega}}{}^{ir}_{rj} = \underset{1}{\overset{\alpha}{\Omega}}{}^{ir}_{si} = 0, \ \underset{1}{\overset{\alpha}{\Omega}}{}^{ir}_{ij} = \tfrac{1}{2}(n-1)\delta^r_j, \ \underset{2}{\overset{\alpha}{\Omega}}{}^{ir}_{ij} = \tfrac{1}{2}(n+1)\delta^r_j, \ (\alpha = 0, 1, 2). \tag{3.9}$$

Theorem 3.2.*([1])The set of all metric N-linear connections with respect to G on the manifold* T^2M *is given by:*

$$\begin{aligned}(3.10)\quad \underset{(\alpha 0)}{L}{}^{i}{}_{jk} &= \underset{(\alpha 0)}{\overset{c}{L}}{}^{i}{}_{jk} + \overset{\alpha}{\underset{1}{\Omega}}{}^{mi}_{jl}\overset{\alpha}{X}{}^{l}{}_{mk}, \quad \underset{(\alpha 1)}{C}{}^{i}{}_{jk} = \underset{(\alpha 1)}{\overset{c}{C}}{}^{i}{}_{jk} + \overset{\alpha}{\underset{1}{\Omega}}{}^{mi}_{jl}\overset{\alpha}{Y}{}^{l}{}_{mk},\\ \underset{(\alpha 2)}{C}{}^{i}{}_{jk} &= \underset{(\alpha 2)}{\overset{c}{C}}{}^{i}{}_{jk} + \overset{\alpha}{\underset{1}{\Omega}}{}^{mi}_{jl}\overset{\alpha}{Z}{}^{l}{}_{mk}, (\alpha = 0,1,2),\end{aligned}$$

where $D\overset{c}{\Gamma}(N) = \left(\underset{(\alpha 0)}{\overset{c}{L}}{}^{i}{}_{jk}, \underset{(\alpha 1)}{\overset{c}{C}}{}^{i}{}_{jk}, \underset{(\alpha 2)}{\overset{c}{C}}{}^{i}{}_{jk}\right)$ *are the local coefficients of the canonical N-linear connection associated with G and* $\overset{\alpha}{X}{}^{i}{}_{jk}, \overset{\alpha}{Y}{}^{i}{}_{jk}, \overset{\alpha}{Z}{}^{i}{}_{jk}$, $(\alpha = 0,1,2)$, *are arbitrary d-tensor fields.*

Definition 3.4. *([1])An N-linear connection on T^2M is called semi-symmetric if:*

$$(3.11)\quad \underset{(00)}{T}{}^{i}{}_{jk} = \tfrac{1}{2}(\delta^i_j\sigma_k - \delta^i_k\sigma_j), \qquad \underset{(\beta\beta)}{S}{}^{i}{}_{jk} = \tfrac{1}{2}(\delta^i_j \underset{(\beta)}{\tau}{}_k - \delta^i_k \underset{(\beta)}{\tau}{}_j),$$

where $\sigma, \underset{(1)}{\tau}, \underset{(2)}{\tau} \in \mathcal{X}^*(T^2M)$ *and* $\underset{(0)}{\overset{0}{T}}{}^{i}{}_{jk} = \underset{(00)}{T}{}^{i}{}_{jk}$, $\underset{(\beta)}{\overset{\beta}{S}}{}^{i}{}_{jk} = \underset{(\beta\beta)}{S}{}^{i}{}_{jk}$, $(\beta = 1,2)$.

Theorem 3.3. *([1]) Let T^2M be endowed with a metric structure G. There exists on T^2M a metric N-linear connection completely determined by G whose h(hh)-, $v_1(v_1v_1)$- and $v_2(v_2v_2)$- tensors of torsion are prescribed and its local coefficients are as follows:*

$$\begin{aligned}(3.12)\quad &\underset{(00)}{L}{}^{i}{}_{jk} = \underset{(00)}{\overset{c}{L}}{}^{i}{}_{jk} + \tfrac{1}{2}\underset{(0)}{g}{}^{im}(\underset{(0)}{g}{}_{mh}\underset{(00)}{T}{}^{h}{}_{jk} - \underset{(0)}{g}{}_{jh}\underset{(00)}{T}{}^{h}{}_{mk} + \underset{(0)}{g}{}_{hk}\underset{(00)}{T}{}^{h}{}_{jm}),\\ &\underset{(\beta 0)}{L}{}^{i}{}_{jk} = \underset{(\beta 0)}{\overset{c}{L}}{}^{i}{}_{jk},\\ &\underset{(\beta\beta)}{C}{}^{i}{}_{jk} = \underset{(\beta\beta)}{\overset{c}{C}}{}^{i}{}_{jk} + \tfrac{1}{2}\underset{(\beta)}{g}{}^{im}(\underset{(\beta)}{g}{}_{mh}\underset{(\beta\beta)}{S}{}^{h}{}_{jk} - \underset{(\beta)}{g}{}_{jh}\underset{(\beta\beta)}{S}{}^{h}{}_{mk} + \underset{(\beta)}{g}{}_{hk}\underset{(\beta\beta)}{S}{}^{h}{}_{jm}),\\ &\underset{(01)}{C}{}^{i}{}_{jk} = \underset{(01)}{\overset{c}{C}}{}^{i}{}_{jk}, \ \underset{(21)}{C}{}^{i}{}_{jk} = \underset{(21)}{\overset{c}{C}}{}^{i}{}_{jk}, \ \underset{(\varepsilon 2)}{C}{}^{i}{}_{jk} = \underset{(\varepsilon 2)}{\overset{c}{C}}{}^{i}{}_{jk}, \ (\varepsilon = 0,1; \beta = 1,2),\end{aligned}$$

where $D\overset{c}{\Gamma}(N) = (\underset{(\alpha 0)}{\overset{c}{L}}{}^{i}{}_{jk}, \underset{(\alpha 1)}{\overset{c}{C}}{}^{i}{}_{jk}, \underset{(\alpha 2)}{\overset{c}{C}}{}^{i}{}_{jk})$, $(\alpha = 0,1,2)$ *are the local coefficients of the canonical N-linear connection associated with G.*

Using the Theorem 3.3 and the Definition 3.4 we obtain:

Theorem 3.4. *The set of all metric semi-symmetric N-linear connections with the local coefficients:* $D\Gamma(N) = (\underset{(\alpha 0)}{L}{}^{i}{}_{jk}, \underset{(\alpha 1)}{C}{}^{i}{}_{jk}, \underset{(\alpha 2)}{C}{}^{i}{}_{jk})$, $(\alpha = 0,1,2)$ *is given by:*

$$\begin{aligned}
&\underset{(00)}{L}{}^{i}_{jk} = \underset{(00)}{\overset{c}{L}}{}^{i}_{jk} + \tfrac{1}{2}(\underset{(0)}{g}{}_{jk}\, \underset{(0)}{g}{}^{im}\sigma_m - \sigma_j\delta^i_k), \quad \underset{(\beta 0)}{L}{}^{i}_{jk} = \underset{(\beta 0)}{\overset{c}{L}}{}^{i}_{jk},\\
(3.13)\quad &\underset{(01)}{C}{}^{i}_{jk} = \underset{(01)}{\overset{c}{C}}{}^{i}_{jk}, \quad \underset{(21)}{C}{}^{i}_{jk} = \underset{(21)}{\overset{c}{C}}{}^{i}_{jk}, \quad \underset{(\varepsilon 2)}{C}{}^{i}_{jk} = \underset{(\varepsilon 2)}{\overset{c}{C}}{}^{i}_{jk},\\
&\underset{(\beta\beta)}{C}{}^{i}_{jk} = \underset{(\beta\beta)}{\overset{c}{C}}{}^{i}_{jk} + \tfrac{1}{2}(\underset{(\beta)}{g}{}_{jk}\, \underset{(\beta)}{g}{}^{im}\, \underset{(\beta)}{\tau}{}_m - \underset{(\beta)}{\tau}{}_j\delta^i_k), \ (\varepsilon = 0,1; \beta = 1,2),
\end{aligned}$$

where $D\overset{c}{\Gamma}(N) = (\underset{(\alpha 0)}{\overset{c}{L}}{}^{i}_{jk}, \underset{(\alpha 1)}{\overset{c}{C}}{}^{i}_{jk}, \underset{(\alpha 2)}{\overset{c}{C}}{}^{i}_{jk})$, $(\alpha = 0,1,2)$ *are the local coefficients of the canonical N-linear connection associated with G and* $\sigma, \underset{(1)}{\tau}, \underset{(2)}{\tau} \in \mathcal{X}^*(T^2M)$.

4 The group of transformations of metric semi-symmetric N-linear connections

Let N be a given nonlinear connection on T^2M. Then any metric semi-symmetric N-linear connection with local coefficients

$$\bar{D}\Gamma(N) = (\underset{(\alpha 0)}{\bar{L}}{}^{i}_{jk}, \underset{(\alpha 1)}{\bar{C}}{}^{i}_{jk}, \underset{(\alpha 2)}{\bar{C}}{}^{i}_{jk}),$$

$(\alpha = 0,1,2)$ is given by (3.12) with (3.11).

From Theorem 3.4 we have:

Theorem 4.1. *Two metric semi-symmetric N-linear connections: D and* $\bar{D}$*, with local coefficients* $D\Gamma(N) = (\underset{(\alpha 0)}{L}{}^{i}_{jk}, \underset{(\alpha 1)}{C}{}^{i}_{jk}, \underset{(\alpha 2)}{C}{}^{i}_{jk})$*, and* $\bar{D}\Gamma(N) = (\underset{(\alpha 0)}{\bar{L}}{}^{i}_{jk}, \underset{(\alpha 1)}{\bar{C}}{}^{i}_{jk}, \underset{(\alpha 2)}{\bar{C}}{}^{i}_{jk})$, $(\alpha = 0,1,2)$ *are related as follows:*

$$\begin{aligned}
&\underset{(00)}{\bar{L}}{}^{i}_{jk} = \underset{(00)}{L}{}^{i}_{jk} + \tfrac{1}{2}(\underset{(0)}{g}{}_{jk}\, \underset{(0)}{g}{}^{im}\sigma_m - \sigma_j\delta^i_k), \quad \underset{(\beta 0)}{\bar{L}}{}^{i}_{jk} = \underset{(\beta 0)}{L}{}^{i}_{jk},\\
(4.1)\quad &\underset{(01)}{\bar{C}}{}^{i}_{jk} = \underset{(01)}{C}{}^{i}_{jk}, \quad \underset{(21)}{\bar{C}}{}^{i}_{jk} = \underset{(21)}{C}{}^{i}_{jk}, \quad \underset{(\varepsilon 2)}{\bar{C}}{}^{i}_{jk} = \underset{(\varepsilon 2)}{C}{}^{i}_{jk},\\
&\underset{(\beta\beta)}{\bar{C}}{}^{i}_{jk} = \underset{(\beta\beta)}{C}{}^{i}_{jk} + \tfrac{1}{2}(\underset{(\beta)}{g}{}_{jk}\, \underset{(\beta)}{g}{}^{im}\, \underset{(\beta)}{\tau}{}_m - \underset{(\beta)}{\tau}{}_j\delta^i_k), \ (\varepsilon = 0,1; \beta = 1,2).
\end{aligned}$$

Conversely, given $\sigma_j \in \mathcal{X}^*(T^2M)$, $\tau_j \in \mathcal{X}^*(T^2M)$ the above (4.1)is thought to be a transformation of a metric semi-symmetric N-linear connection D, with the local coefficients $D\Gamma(N) = (\underset{(\alpha 0)}{L}{}^{i}_{jk}, \underset{(\alpha 1)}{C}{}^{i}_{jk}, \underset{(\alpha 2)}{C}{}^{i}_{jk})$, to a metric semi-symmetric N-linear connection $\bar{D}$, with the local coefficients $\bar{D}\Gamma(N) = (\underset{(\alpha 0)}{\bar{L}}{}^{i}_{jk}, \underset{(\alpha 1)}{\bar{C}}{}^{i}_{jk}, \underset{(\alpha 2)}{\bar{C}}{}^{i}_{jk})$, $(\alpha = 0,1,2)$.

We shall denote this transformation by: $t(\sigma_j, \underset{(1)}{\tau}{}_j, \underset{(2)}{\tau}{}_j)$.

Thus we have:

Theorem 4.2. *The set:* $\overset{ms}{\mathcal{T}}_N$ *of all transformations* $t(\sigma, \underset{(1)}{\tau}, \underset{(2)}{\tau}) : D\Gamma(N) \to \bar{D}\Gamma(N)$ *of the metric semi-symmetric N-linear connections, given by (4.1) is an Abelian group, together with the mapping product.*

This group acts on the set of all metric semi-symmetric N-linear connections, corresponding to the same nonlinear connection N, transitively.

By applying the results from Proposition 2.3, we obtain:

Theorem 4.3. *By means of a transformation (4.1) the tensor fields* $\underset{(00)}{\mathbb{K}}{}_h{}^i{}_{jk}$, $\underset{(\beta\beta)}{\mathbb{S}}{}_h{}^i{}_{jk}$, $(\beta = 1, 2)$ *given in (2.8), (2.12) are changed by the laws:*

$$\underset{(00)}{\bar{\mathbb{K}}}{}_h{}^i{}_{jk} = \underset{(00)}{\mathbb{K}}{}_h{}^i{}_{jk} + \mathcal{A}_{jk}\{\overset{0}{\underset{1}{\Omega}}{}^{ir}_{jh}\sigma_{rk}\}, \tag{4.2}$$

$$\underset{(\beta\beta)}{\bar{\mathbb{S}}}{}_h{}^i{}_{jk} = \underset{(\beta\beta)}{\mathbb{S}}{}_h{}^i{}_{jk} + \mathcal{A}_{jk}\{\overset{\beta}{\underset{1}{\Omega}}{}^{ir}_{jh}\underset{(\beta)}{\tau}{}_{rk}\}, \tag{4.3}$$

where:

$$\sigma_{rk} = -\sigma_{\underset{r|k}{0}} + \tfrac{1}{2}\sigma_r\sigma_k + \tfrac{1}{4}\underset{(0)}{g}{}_{rk}\sigma, \ (\sigma = \underset{(0)}{g}{}^{rm}\sigma_r\sigma_m), \tag{4.4}$$

$$\underset{(\beta)}{\tau}{}_{rk} = -\underset{(\beta)}{\tau}{}_{\underset{r|\beta k}{\beta}} + \tfrac{1}{2}\underset{(\beta)}{\tau}{}_r\underset{(\beta)}{\tau}{}_k + \tfrac{1}{4}\underset{(\beta)}{g}{}_{rk}\underset{(\beta)}{\tau}, \ (\underset{(\beta)}{\tau} = \underset{(\beta)}{g}{}^{rm}\underset{(\beta)}{\tau}{}_r\underset{(\beta)}{\tau}{}_k), \ (\beta = 1, 2). \tag{4.5}$$

Using these results we can determine some invariants of the group $\overset{ms}{\mathcal{T}}_N$. To this aim we eliminate σ_{ij}, $\underset{(\beta)}{\tau}{}_{ij}$, $(\beta = 1, 2)$ from (4.2), (4.3) and we obtain:

Theorem 4.4. *For* $n > 2$ *the following tensor fields* $\underset{(00)}{H}{}_h{}^i{}_{jk}$, $\underset{(\beta\beta)}{M}{}_h{}^i{}_{jk}$, $(\beta = 1, 2)$ *of metric semi-symmetric N-linear connections on* T^2M*, are invariants of the group* $\overset{ms}{\mathcal{T}}_N$:

$$\underset{(00)}{H}{}_h{}^i{}_{jk} = \underset{(00)}{\mathbb{K}}{}_h{}^i{}_{jk} + \tfrac{1}{n-2}\mathcal{A}_{jk}\{\overset{0}{\underset{1}{\Omega}}{}^{ir}_{jh}(2\underset{(00)}{\mathbb{K}}{}_{rk} - \frac{\underset{(00)}{\mathbb{K}}\underset{(0)}{g}{}_{rk}}{n-1})\}, \tag{4.6}$$

$$\underset{(\beta\beta)}{M}{}_h{}^i{}_{jk} = \underset{(\beta\beta)}{\mathbb{S}}{}_h{}^i{}_{jk} + \tfrac{1}{n-2}\mathcal{A}_{jk}\{\overset{\beta}{\underset{1}{\Omega}}{}^{ir}_{jh}(2\underset{(\beta\beta)}{\mathbb{S}}{}_{rk} - \frac{\underset{(\beta\beta)}{\mathbb{S}}\underset{(\beta)}{g}{}_{rk}}{n-1})\}, \tag{4.7}$$

where: $\underset{(00)}{\mathbb{K}}{}_{hj} = \underset{(00)}{\mathbb{K}}{}_h{}^i{}_{ji}$, $\underset{(00)}{\mathbb{K}} = \underset{(0)}{g}{}^{hj}\underset{(00)}{\mathbb{K}}{}_{hj}$, $\underset{(\beta\beta)}{\mathbb{S}}{}_{hj} = \underset{(\beta\beta)}{\mathbb{S}}{}_h{}^i{}_{ji}$, $\underset{(\beta\beta)}{\mathbb{S}} = \underset{(\beta)}{g}{}^{hj}\underset{(\beta\beta)}{\mathbb{S}}{}_{hj}$,

$(\beta = 1, 2)$.

In order to find other invariants of the group $\overset{ms}{\mathcal{T}}_N$, let us consider the transformation formulas of the torsion d-tensor fields by a transformation $t(\sigma, \underset{(1)}{\tau}, \underset{(2)}{\tau}) : D\Gamma(N) \to \bar{D}\Gamma(N)$ of metric semi-symmetric N-linear connections on T^2M, with respect to G, given by (4.1)

Using Proposition 2.1. and the transformation (4.1) by direct calculations we obtain:

Proposition 4.1. *By a transformation (4.1) of metric semi-symmetric N-linear connections, corresponding to the same nonlinear connection N:* $t(\sigma_j, \underset{(1)}{\tau}{}_j, \underset{(2)}{\tau}{}_j) : D\Gamma(N) \to \bar{D}\Gamma(N)$, *the torsion tensor fields,*

$\underset{(00)}{T}{}^i_{jk}$, $\underset{(0\beta)}{R}{}^i_{jk}$, $\underset{(\beta 0)}{T}{}^i_{jk}$, $\underset{(\beta\beta)}{S}{}^i_{jk}$, $\underset{(\alpha\beta)}{S}{}^i_{jk}$, $\underset{(21)}{Q}{}^i_{jk}$, $\underset{(\beta\beta)}{P}{}^i_{jk}$, $\underset{(\beta 0)}{P}{}^i_{jk}$, $\underset{(12)}{P}{}^i_{jk}$, $\underset{(21)}{P}{}^i_{jk}$, *are transformed as follows*:

$$
\begin{array}{ll}
 & \underset{(0\beta)}{\bar{R}}{}^i_{jk} = \underset{(0\beta)}{R}{}^i_{jk}, \quad \underset{(00)}{\bar{T}}{}^i_{jk} = \underset{(00)}{T}{}^i_{jk} + \mathcal{A}_{ij}\{\sigma_j \delta^i_k\}, \quad \underset{(\beta 0)}{\bar{T}}{}^i_{jk} = \underset{(\beta 0)}{T}{}^i_{jk}, \\
(4.8) & \underset{(\beta\beta)}{\bar{S}}{}^i_{jk} = \underset{(\beta\beta)}{S}{}^i_{jk} + \mathcal{A}_{ij}\{\underset{(\beta)}{\tau}{}_j \delta^i_k\}, \quad \underset{(\alpha\beta)}{\bar{S}}{}^i_{jk} = \underset{(\alpha\beta)}{S}{}^i_{jk}, \quad \underset{(21)}{\bar{Q}}{}^i_{jk} = \underset{(21)}{Q}{}^i_{jk}, \\
 & \underset{(\beta\beta)}{\bar{P}}{}^i_{jk} = \underset{(\beta\beta)}{P}{}^i_{jk}, \quad \underset{(\beta 0)}{\bar{P}}{}^i_{jk} = \underset{(\beta 0)}{P}{}^i_{jk}, \quad \underset{(12)}{\bar{P}}{}^i_{jk} = \underset{(12)}{P}{}^i_{jk}, \quad \underset{(21)}{\bar{P}}{}^i_{jk} = \underset{(21)}{P}{}^i_{jk} \\
 & (\alpha, \beta = 1, 2; \alpha \neq \beta).
\end{array}
$$

We denote with:

$$
(4.9) \qquad \underset{(\beta)}{t}{}^i_{jk} = \mathcal{A}_{jk}\left\{\frac{\delta \underset{\beta}{N}{}^i_j}{\delta y^{(\beta)k}}\right\}, \ (\beta = 1, 2),
$$

and with:

$$(4.10)\quad \begin{aligned}
&\underset{(\beta)}{t}{}^{*}_{ijk}=\Sigma_{ijk}\{\underset{(\beta)}{g}{}_{im}\underset{(\beta)}{t}{}^{m}_{jk}\},\ \ \underset{(00)}{T}{}^{*}_{ijk}=\Sigma_{ijk}\{\underset{(0)}{g}{}_{im}\underset{(00)}{T}{}^{m}_{jk}\},\\
&\underset{(\beta 0)}{T}{}^{*}_{ijk}=\Sigma_{ijk}\{\underset{(\beta)}{g}{}_{im}\underset{(\beta 0)}{T}{}^{m}_{jk}\},\\
&\underset{(0\beta)}{R}{}^{*}_{ijk}=\Sigma_{ijk}\{\underset{(0)}{g}{}_{im}\underset{(0\beta)}{R}{}^{m}_{jk}\},\ \ \underset{(\beta\beta)}{S}{}^{*}_{ijk}=\Sigma_{ijk}\{\underset{(\beta)}{g}{}_{im}\underset{(\beta\beta)}{S}{}^{m}_{jk}\},\\
&\underset{(\alpha\beta)}{S}{}^{*}_{ijk}=\Sigma_{ijk}\{\underset{(\alpha)}{g}{}_{im}\underset{(\alpha\beta)}{S}{}^{m}_{jk}\},\\
&\underset{(\beta\beta)}{C}{}^{*}_{ijk}=\Sigma_{ijk}\{\underset{(\beta)}{g}{}_{im}\underset{(\beta\beta)}{C}{}^{m}_{jk}\},\ \ \underset{(00)}{L}{}^{*}_{ijk}=\Sigma_{ijk}\{\underset{(0)}{g}{}_{im}\underset{(00)}{L}{}^{m}_{jk}\},\\
&\underset{(\beta\beta)}{P}{}^{*}_{ijk}=\Sigma_{ijk}\{\underset{(\beta)}{g}{}_{im}\underset{(\beta\beta)}{P}{}^{m}_{jk}\},\\
&\underset{(\alpha\beta)}{P}{}^{*}_{ijk}=\Sigma_{ijk}\{\underset{(\alpha)}{g}{}_{im}\underset{(\alpha\beta)}{P}{}^{m}_{jk}\},\ \ \underset{(\beta 0)}{P}{}^{*}_{ijk}=\Sigma_{ijk}\{\underset{(\beta)}{g}{}_{im}\underset{(\beta 0)}{P}{}^{m}_{jk}\},\\
&\underset{(21)}{Q}{}^{*}_{ijk}=\Sigma_{ijk}\{\underset{(2)}{g}{}_{im}\underset{(21)}{Q}{}^{m}_{jk}\},\ (\alpha,\beta=1,2;\alpha\neq\beta),
\end{aligned}$$

where $\Sigma_{ijk}\{...\}$ *denotes the cyclic summation, and with:*

$$(4.11)\quad \begin{aligned}
&\underset{(00)}{\overset{1}{\mathcal{K}}}{}_{ijk}=-\underset{(0)}{g}{}_{km}\underset{(00)}{T}{}^{m}_{ij}+\mathcal{A}_{ij}\{\underset{(0)}{g}{}_{im}\underset{(00)}{L}{}^{m}_{jk}\},\\
&\underset{(\beta\beta)}{\overset{2}{\mathcal{K}}}{}_{ijk}=\underset{(\beta)}{g}{}_{im}\underset{(\beta\beta)}{S}{}^{m}_{jk}+2\mathcal{A}_{jk}\{\underset{(\beta)}{g}{}_{km}\underset{(\beta\beta)}{C}{}^{m}_{ij}\},\quad \underset{(10)}{\overset{3}{\mathcal{K}}}{}_{ijk}=\mathcal{A}_{jk}\{\underset{(1)}{g}{}_{km}\underset{(11)}{P}{}^{m}_{ij}\},\\
&\underset{(\alpha\beta)}{\overset{3}{\mathcal{K}}}{}_{ijk}=\mathcal{A}_{jk}\{\underset{(\beta)}{g}{}_{km}\underset{(\alpha\beta)}{P}{}^{m}_{ij}\},\quad \underset{(\beta\beta)}{\overset{4}{\mathcal{K}}}{}_{ijk}=\underset{(\beta)}{g}{}_{mj}\underset{(\beta\beta)}{C}{}^{m}_{ik}+\underset{(\beta)}{g}{}_{im}\underset{(\beta\beta)}{C}{}^{m}_{jk},\\
&\overset{1}{\mathcal{S}}{}_{ijk}=-\underset{(2)}{g}{}_{jm}\underset{(22)}{P}{}^{m}_{ik}-\underset{(1)}{g}{}_{mk}\underset{(11)}{P}{}^{m}_{ij},\quad \underset{(21)}{\overset{2}{\mathcal{S}}}{}_{ijk}=\mathcal{A}_{jk}\{\underset{(2)}{g}{}_{mj}\underset{(21)}{Q}{}^{m}_{ik}\},\\
&\underset{(\alpha\beta)}{\overset{3}{\mathcal{S}}}{}_{ijk}=\mathcal{A}_{ij}\{\underset{(\alpha)}{g}{}_{im}\underset{(\alpha\beta)}{P}{}^{m}_{jk}\},\ (\alpha,\beta=1,2;\alpha\neq\beta).
\end{aligned}$$

Remark 4.1. *It is noted that:* $\underset{(\beta)}{t}{}^{*}_{ijk}$, $\underset{(00)}{T}{}^{*}_{ijk}$, $\underset{(10)}{T}{}^{*}_{ijk}$, $\underset{(0\beta)}{R}{}^{*}_{ijk}$, $\underset{(\beta\beta)}{S}{}^{*}_{ijk}$, $\underset{(12)}{R}{}^{*}_{ijk}$, $(\beta=1,2)$ *are alternate,* $\underset{(00)}{\overset{1}{\mathcal{K}}}{}_{ijk}$, $\underset{(\alpha\beta)}{\overset{3}{\mathcal{S}}}{}_{ijk}$, $(\alpha,\beta=1,2;\alpha\neq\beta)$, *are alternate, with respect to: i,j and* $\underset{(\beta\beta)}{\overset{2}{\mathcal{K}}}{}_{ijk}$, $\underset{(\alpha\beta)}{\overset{3}{\mathcal{K}}}{}_{ijk}$, $\underset{(21)}{\overset{2}{\mathcal{S}}}{}_{ijk}$, $(\alpha,\beta=1,2;\alpha\neq\beta)$, *are alternate with respect to: j,k.*

Theorem 4.5. *The tensor fields:* $\underset{(\beta)}{t}{}^{i}_{jk}$, $\underset{(0\beta)}{R}{}^{i}_{jk}$, $\underset{(\beta 0)}{T}{}^{i}_{jk}$, $\underset{(\alpha\beta)}{S}{}^{i}_{jk}$, $\underset{(21)}{Q}{}^{i}_{jk}$, $\underset{(\beta\beta)}{P}{}^{i}_{jk}$, $\underset{(\beta 0)}{P}{}^{i}_{jk}$, $\underset{(\alpha\beta)}{P}{}^{i}_{jk}$, $\underset{(\beta)}{t}{}^{*}_{ijk}$, $\underset{(00)}{T}{}^{*}_{ijk}$, $\underset{(\beta 0)}{T}{}^{*}_{ijk}$, $\underset{(00)}{L}{}^{*}_{ijk}$, $\underset{(\alpha\beta)}{S}{}^{*}_{ijk}$, $\underset{(\beta\beta)}{S}{}^{*}_{ijk}$, $\underset{(12)}{R}{}^{*}_{ijk}$, $\underset{(0\beta)}{R}{}^{*}_{ijk}$, $\underset{(\beta\beta)}{C}{}^{*}_{ijk}$, $\underset{(21)}{Q}{}^{*}_{ijk}$, $\underset{(\alpha\beta)}{P}{}^{*}_{ijk}$, $\underset{(\beta\beta)}{P}{}^{*}_{ijk}$, $\underset{(\beta 0)}{P}{}^{*}_{ijk}$, $\underset{(00)}{\overset{1}{\mathcal{K}}}{}_{ijk}$, $\underset{(\beta\beta)}{\overset{2}{\mathcal{K}}}{}_{ijk}$, $\underset{(\alpha\beta)}{\overset{3}{\mathcal{K}}}{}_{ijk}$, $\underset{(10)}{\overset{3}{\mathcal{K}}}{}_{ijk}$, $\underset{(\beta\beta)}{\overset{4}{\mathcal{K}}}{}_{ijk}$, $\overset{1}{\mathcal{S}}{}_{ijk}$, $\underset{(21)}{\overset{2}{\mathcal{S}}}{}_{ijk}$,

$\overset{3}{\underset{(\alpha\beta)}{\mathcal{S}}}{}_{ijk}$, $(\alpha, \beta = 1, 2; \alpha \neq \beta)$ *are invariants of the group* $\overset{ms}{\mathcal{T}}_N$.

Proof. By means of transformations of the torsion given in (4.8) and using the notations (4.9), (4.10), (4.11), by direct calculation from (4.1) we have: $\underset{(00)}{\bar{T}}{}^{*}_{ijk} = \underset{(00)}{T}{}^{*}_{ijk}$ etc.

Theorem 4.6. *Between the invariants in Theorem 4.5 there exists the following relations:*

$$(4.12)\quad \begin{aligned} &\Sigma_{ijk} \overset{1}{\underset{(00)}{\mathcal{K}}}{}_{ijk} = 0,\ \Sigma_{ijk} \overset{2}{\underset{(\beta\beta)}{\mathcal{K}}}{}_{ijk} = 3 \underset{(\beta\beta)}{S}{}^{*}_{ijk},\ \Sigma_{ijk} \overset{3}{\underset{(10)}{\mathcal{K}}}{}_{ijk} = \underset{(10)}{T}{}^{*}_{ijk} + \underset{(1)}{t}{}^{*}_{ijk}, \\ &\Sigma_{ijk} \overset{3}{\underset{(\alpha\beta)}{\mathcal{K}}}{}_{ijk} = \underset{(\alpha\beta)}{P}{}^{*}_{ijk} - \underset{(\alpha\beta)}{P}{}^{*}_{ikj}, \\ &\Sigma_{ijk} \overset{4}{\underset{(\beta\beta)}{\mathcal{K}}}{}_{ijk} = \underset{(\beta\beta)}{C}{}^{*}_{ijk} - \underset{(\beta\beta)}{C}{}^{*}_{ikj},\ \mathcal{A}_{ij}\{ \overset{4}{\underset{(\beta\beta)}{\mathcal{K}}}{}_{ijk}\} = 0, \\ &\Sigma_{ijk} \overset{1}{\mathcal{S}}{}_{ijk} = - \overset{3}{\underset{(10)}{K}}{}_{ijk} - \Sigma_{ijk} \underset{(2)}{g}{}_{jm} \underset{(22)}{P}{}^{m}_{ik}, \quad \Sigma_{ijk} \overset{2}{\underset{(21)}{\mathcal{S}}}{}_{ijk} = -\mathcal{A}_{jk}\{ \underset{(21)}{Q}{}^{*}_{ijk}\}, \\ &\Sigma_{ijk} \overset{3}{\underset{(\alpha\beta)}{\mathcal{S}}}{}_{ijk} = \mathcal{A}_{ij}\{ \underset{(\alpha\beta)}{P}{}^{*}_{ijk}\},\ (\alpha, \beta = 1, 2; \alpha \neq \beta). \end{aligned}$$

Proof. Using the notations (4.9), (4.10), (4.11), the Remark 4.1 and the definitions of the torsion d-tensor fields given in [1], by direct calculations we obtain the results.

References

[1] Atanasiu, GH., New Aspects in Differential Geometry of the Second Order, Sem.de Mecanică, Univ. de Vest dinTimişoara, No.82, 2001, 1-81.

[2] Atanasiu, GH., Linear Connections in the Differential Geometry of Order Two, in the volume: Lagrange and Hamilton Geometries and Their Applications, ed. R.Miron, Fair Partners Publ., Bucureşti,49, 2004, 11-30.

[3] Atanasiu, GH., Calculus with N-and JN-linear connections on the tangent bundle of the order two. Torsions and curvatures (to appear).

[4] Miron, R., The Geometry of Higher Order Lagrange Spaces. Appli-

cations to Mechanics and Physics, Kluwer Acad. Publ., FTPH no.82, 1997.

[5] Miron, R. and Atanasiu, GH., Compendium on the higher-order Lagrange spaces: The geometry of k-osculator bundle. Prolongation of the Riemannian, Finslerian and Lagrangian structures. Lagrange spaces, Tensor, N.S., 53, 1993, 39-57.

[6] Miron, R. and Atanasiu, GH., Compendium sur les espaces Lagrange d'ordre supérieur: La géométrie du fibré k-osculateur; Le prolongement des structures Riemanniennes, Finsleriennes et Lagrangiennes; Les espaces Lagrange $L^{(k)n}$, Sem. de Mecanică, Univ. de Vest din Timişoara, No.40, 1994, 1-27.

[7] Miron, R. and Atanasiu, GH., Lagrange Geometry of Second Order, Matt. Comput. Modelling, Pergamon, 20, 4/5, 1994, 41-56.

[8] Miron, R. and Atanasiu, GH., Differential Geometry of the k-Osculator Bundle, Rev. Roumaine Math. Pures et Appl., 41, 3/4, 1996, 205-236.

[9] Obădeanu, V., Structures géométriques associées à certains systèmes dynamiques, Sem. de Mecanică, Univ. de Vest din Timişoara, No.67, 2000, 1-11.

[10] Olver, P.J., Applications of Lie Groups to Differential Equations, Graduate Texts in Mathematics, vol. 107, Springer-Verlag, New York, 1986.

[11] Oproiu, V., Some remarkable structures and connections defined on the tangent bundle, Rendiconti di Matematica, 6(3), 1973, 503-540.

[12] Purcaru, M., On Transformations Groups of N-Linear Connections in the Bundle of Accelerations, General Mathematics, 5, 1997, 339-346.

[13] Purcaru, M., Remarkable Geometrical Structures in the Second Order Tangent Bundle (in Romannian), Ph.D.Thesis, Univ. Babeş-Bolyai, Cluj-Napoca, 2002, 1-246.

[14] Puta, M., Hamiltonian Mechanical Systems and Geometric Quantization, Kluwer Acad. Publ., 260, 1993.

[15] Voicu, N., Deviations of Geodesics in the Geometry of the Order Two (in Romannian), Ph.D.Thesis, Univ. Babeş-Bolyai, Cluj-Napoca, 2003, 1-134.

[16] Yano, K., and Ishihara, S., Tangent and Cotangent Bundles. Differential Geometry, M. Dekker, Inc., New York, 1973.

Address:

Department of Algebra and Geometry
"Transilvania" University
50, Iuliu Maniu Street
500091 Braşov, România

E-mail:
Gheorghe Atanasiu: g.atanasiu@unitbv.ro gh_atanasiu@yahoo.com
Monica Purcaru: mpurcaru@unitbv.ro