Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **21:2** (2007), 1–9

# SOME SUBCLASSES OF MEROMORPHIC FUNCTIONS

**Mugur Acu**

**Abstract**

In this paper we define some subclass of meromorphic functions and we obtain some properties of these classes.

## 1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, $\mathcal{H}_u(U) = \{f \in \mathcal{H}(U) : f\, is\, univalent\, in\, U\}$ and $S = \{f \in A : f\, is\, univalent\, in\, U\}$. Let $\dot{U} = \{w \in \mathbb{C} : 0 < |w| < 1\}$ , $U^- = \{\zeta \in \mathbb{C}_\infty : |\zeta| > 1\}$ and $\Sigma$ be the class of all functions $\varphi$ which are meromorphic in $U^-$ with $\varphi(\infty) = \infty$ and $\varphi'(\infty) = 1$:

$$\varphi(\zeta) = \zeta + \alpha_0 + \frac{\alpha_1}{\zeta} + \cdots + \frac{\alpha_n}{\zeta^n} + \cdots \ , \ |\zeta| > 1 \tag{1}$$

The purpose of this note is to define some subclasses of close to convex meromorphic functions and give some properties of these classes.

## 2 Preliminary results

Let $E(\varphi) = \mathbb{C} \setminus \varphi(U^-)$ and $\Sigma_0 = \{\varphi \in \Sigma : \varphi(\zeta) \neq 0 \ , \ \zeta \in U^-\}$.

2000 *Mathematics Subject Classifications.* 30C45.
*Key words and Phrases.* Meromorphic functions, integral operators.
Received: May 5, 2004

**Remark 2.1** *We have the bijection* $T : S \to \Sigma_0$ *with*
$T(f) = \varphi\ ,\ \varphi(\zeta) = \dfrac{1}{f(z)}\ ,\ \zeta = \dfrac{1}{z}\ ,\ \zeta \in U^-$ *and* $T(S) = \Sigma_0$,
$T^{-1}(\Sigma_0) = S$. *It is easy to see that* $\dfrac{\zeta\varphi'(\zeta)}{\varphi(\zeta)} = \dfrac{zf'(z)}{f(z)}\ ,\ \zeta = \dfrac{1}{z}\ ,\ \zeta \in U^-$.

**Definition 2.1** *[4] A function* $\varphi$ *in form (1) is called starlike in* $U^-$ *if* $\varphi$ *is univalent in* $U^-$ *and* $E(\varphi)$ *is starlike in respect to the origin.*

**Remark 2.2** *From Definition 2.1 we have* $0 \in E(\varphi)$ *and thus* $\varphi \in \Sigma_0$.

**Definition 2.2** *[4]* $\Sigma^* = \{\varphi \in \Sigma_0 : \varphi\, is\, starlike\, in\, U^-\}$.

**Remark 2.3** *From Remark 2.1 we have* $\Sigma^* = \{\varphi \in \Sigma_0 : Re\dfrac{\zeta\varphi'(\zeta)}{\varphi(\zeta)} > 0\ ,\ \zeta \in U^-\}$ *and* $\Sigma^* = T(S^*)$ , *where* $S^*$ *is the well know class of starlike functions. If* $\gamma_\rho = \{\zeta : |\zeta| = \rho > 1\}$ *and we denote* $\Gamma_\rho = \varphi(\gamma_\rho)$ *then the condition* $Re\dfrac{\zeta\varphi'(\zeta)}{\varphi(\zeta)} > 0\ ,\ \zeta \in U^-$ *show that if* $\varphi$ *is starlike in* $U^-$ *then* $\Gamma_\rho$ *is a starlike curve in respect to the origin for all* $\rho > 1$.

**Definition 2.3** *[4] Let* $g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots\ ,\ 0 < |z| < 1$ *meromorphic in* $\dot{U}$. *We say that* $g$ *is starlike in* $\dot{U}$ *if the function* $\varphi(\zeta) = g(\frac{1}{\zeta})\ ,\ \zeta \in U^-$ *is starlike in* $U^-$.

**Remark 2.4** *For* $g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots\ ,\ z \in U$ , $g$ *univalent in* $\dot{U}$ , $g(z) \neq 0\ ,\ z \in \dot{U}$ *and* $\varphi(\frac{1}{z}) = g(z)$ *with* $\varphi \in \Sigma_0$ *we have*

$$\frac{zg'(z)}{g(z)} = -\frac{\zeta\varphi'(\zeta)}{\varphi(\zeta)}\ ,\ \zeta = \frac{1}{z},\quad z \in U.$$

**Theorem 2.1** *[4] Let* $g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots\ ,\ z \in U$ *meromorphic in* $U$ *with* $g(z) \neq 0\ ,\ z \in \dot{U}$. *Then* $g$ *is starlike in* $U$ *if and only if* $g$ *is univalent in* $\dot{U}$ *and* $Re\left[-\dfrac{zg'(z)}{g(z)}\right] > 0\ ,\ z \in \dot{U}$.

*We denote also with* $\Sigma^*$ *the class of all these functions.*

**Definition 2.4** *[4] Let* $\alpha < 1$. *We say that a function* $f \in \Sigma$ *is a meromorphic starlike of order* $\alpha$ *function if* $Re\left[-\dfrac{zf'(z)}{f(z)}\right] > \alpha\ ,\ z \in \dot{U}$. *We denote this class with* $\Sigma^*(\alpha)$.

Let $\gamma \in \mathbb{C}$ with $Re\,\gamma > 0$. We consider the integral operator $I_\gamma : \Sigma \to \Sigma$ defined as

$$I_\gamma(F)(z) = \frac{\gamma}{z^{\gamma+1}} \int_0^z t^\gamma \cdot F(t)dt\,. \tag{2}$$

**Theorem 2.2** *[5] Let $\alpha < 1$ , $\gamma > 0$ and $F \in \Sigma^*(\alpha)$. If $f = I_\gamma(F)$, where $I_\gamma$ is defined in (2), with $f(z) \neq 0$ , $z \in \dot{U}$, then $f \in \Sigma^*(\beta)$ where*

$$\beta = \frac{1}{4}\left[2\alpha + 2\gamma + 3 - \sqrt{(2(\gamma-\alpha)+1)^2 + 8\gamma}\right].$$

**Remark 2.5** *It is easy to see that from Theorem 2.2 , with $\alpha = 0$, we have for $F \in \Sigma^*$ and $f = I_\gamma(F)$, with $f(z) \neq 0$ , $z \in \dot{U}$, that $f \in \Sigma^*$.*

The next theorem is due to P.T. Mocanu and S.S. Miller (see [1] , [2] , [3]).

**Theorem 2.3** *Let $q$ convex in $U$ and $j : U \to \mathbb{C}$ with $Re\,[j(z)] > 0$. If $p \in \mathcal{H}(U)$ and $p$ satisfied $p(z) + j(z) \cdot zp'(z) \prec q(z)$ then $p(z) \prec q(z)$.*

# 3 Main results

**Definition 3.1** *Let $\varphi \in \Sigma_0$. We say that $\varphi$ is close to convex in $U^-$ if subsist a function $\psi \in \Sigma^*$ such as $Re\dfrac{\zeta\varphi'(\zeta)}{\psi(\zeta)} > 0$ , $\zeta \in U^-$. We denote with $\Sigma^{cc}$ this class. Thus*

$$\Sigma^{cc} = \left\{\varphi \in \Sigma_0 : (\exists)\, \psi \in \Sigma^*\,,\ Re\frac{\zeta\varphi'(\zeta)}{\psi(\zeta)} > 0\,,\ \zeta \in U^-\right\}$$

**Remark 3.1** *If we take $\varphi \in \Sigma^*$ , $\varphi = \psi$, we have $\Sigma^* = \Sigma^{cc}$.*

**Definition 3.2** *Let $f(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots$ , $0 < |z| < 1$ meromorphic in $\dot{U}$. We say that $f$ is close to convex in $\dot{U}$ with respect to the function $g(z) = \frac{1}{z} + \beta_0 + \beta_1 z + \cdots$ , $0 < |z| < 1$ meromorphic and starlike in $\dot{U}$, if the function $\varphi(\zeta) = f\left(\frac{1}{\zeta}\right)$ , $\zeta \in U^-$ is close to convex in $U^-$ with respect to the function $\psi(\zeta) = g\left(\frac{1}{\zeta}\right)$ , $\zeta \in U^-$ which is starlike in $U^-$.*

**Theorem 3.1** *Let $f(z)=\frac{1}{z}+\alpha_0+\alpha_1 z+\cdots$ , $z\in U$ meromorphic in $U$ with $f(z)\neq 0$ , $z\in\dot{U}$. Then the function $f$ is close to convex in $U$ with respect to the function $g(z)=\frac{1}{z}+\beta_0+\beta_1 z+\cdots$ , $z\in U$, with $g(z)\neq 0$ , $z\in\dot{U}$, meromorphic and starlike in $U$, if and only if $f$ is univalent in $\dot{U}$ and $Re\left[-\frac{zf'(z)}{g(z)}\right]>0$ , $z\in\dot{U}$.*

*We denote also with $\Sigma^{cc}$ the class of all this functions.*

*Proof.*

We know that the conditions $f$ is univalent in $\dot{U}$ and $f(z)\neq 0$ , $z\in\dot{U}$ are equivalent with $\varphi\in\Sigma_0$ where $\varphi\left(\frac{1}{z}\right)=f(z)=\frac{1}{z}+\alpha_0+\alpha_1 z+\cdots$

From Theorem 2.1 with $g(z)=\frac{1}{z}+\beta_0+\beta_1 z+\cdots$, $g(z)\neq 0$ , $z\in\dot{U}$ which is meromorphic and starlike in $U$, we have $\psi\in\Sigma^*$ where $\psi\left(\frac{1}{z}\right)=g(z)$.

It is easy to see that

$$\frac{\zeta\varphi'(\zeta)}{\psi(\zeta)}=-\frac{zf'(z)}{g(z)}\ ,\ \zeta=\frac{1}{z}\ ,\ z\in U\ .$$

Thus f is close to convex in $\dot{U}$ with respect to the function $g$ starlike in $\dot{U}$ if and only if $Re\left[-\frac{zf'(z)}{g(z)}\right]>0$ , $z\in\dot{U}$.

**Definition 3.3** *Let $0\le\alpha<1$ and $\beta>\alpha$. we say that a function $g\in\Sigma$ is in the class $\Sigma^*_{\alpha,\beta}$ if $\alpha<Re\left[-\frac{zg'(z)}{g(z)}\right]<\beta$ , $z\in\dot{U}$. Thus*

$$\Sigma^*_{\alpha,\beta}=\left\{g\in\Sigma\ :\ \alpha<Re\left[-\frac{zg'(z)}{g(z)}\right]<\beta\ ,\ z\in\dot{U}\right\}\ .$$

**Remark 3.2** *It is easy to see that $\Sigma^*_{\alpha,\beta}\subset\Sigma^*(\alpha)\subset\Sigma^*$ , and if we consider $\beta\to\infty$ in the definition of the class $\Sigma^*_{\alpha,\beta}$ we obtain the definition of the class $\Sigma^*(\alpha)$.*

**Theorem 3.2** *Let $0\le\alpha<1$ , $\beta>\alpha$ , $\gamma>0$ and $G(z)\in\Sigma^*_{\alpha,\beta}$. If $g(z)=I_\gamma(G)(z)$, where $I_\gamma$ is defined in (2), and $g(z)\neq 0$ , $z\in\dot{U}$, then $g\in\Sigma^*_{\alpha,\beta'}$ where $\beta'=min\{\beta,\gamma+1\}$.*

*Proof.*

In the conditions from hypothesis it is sufficient to prove that

$$\alpha < Re\left\{-\frac{zg'(z)}{g(z)}\right\} < \beta',$$

where $\beta' = min\{\beta, \gamma+1\}$, $z \in \dot{U}$ .

From (2) we have:

$$(\gamma+1)g(z) + zg'(z) = \gamma G(z) \tag{3}$$

Then from this relation we obtain:

$$(\gamma+2)g'(z) + zg''(z) = \gamma G'(z) \tag{4}$$

We observe that from Remark 2.5 we have $g \in \Sigma^*$ i.e.

$$Re\left[-\frac{zg'(z)}{g(z)}\right] > 0\ ,\ z \in \dot{U}.$$

We have from (3) and (4):

$$-\frac{zG'(z)}{G(z)} = -\frac{z\gamma G'(z)}{\gamma G(z)} = \frac{(\gamma+2)\cdot p(z) - \dfrac{z^2 g''(z)}{g(z)}}{(\gamma+1) - p(z)} \tag{5}$$

where $p(z) = -\dfrac{zg'(z)}{g(z)}$ with $p \in \mathcal{H}(U)$ and $p(0) = 1$.

From $p(z) = -\dfrac{zg'(z)}{g(z)}$ we obtain:

$$zp'(z) = p(z) - \frac{z^2 g''(z)}{g(z)} + p^2(z)$$

and thus

$$-\frac{z^2 g''(z)}{g(z)} = zp'(z) - p(z)(p(z)+1).$$

In this conditions from (5) we have:

$$-\frac{zG'(z)}{G(z)} = p(z) + \frac{1}{(\gamma+1) - p(z)} \cdot zp'(z).$$

Let $q$ convex in $U$ with $q(U) = \Delta = \{w \in \mathbb{C} : \alpha < Re\, w < \beta\}$. Then from $G \in \Sigma^*_{\alpha,\beta}$ we have $-\dfrac{zG'(z)}{G(z)} \prec q(z)$ and thus:

$$p(z) + \frac{1}{(\gamma+1) - p(z)} \cdot zp'(z) \prec q(z)$$

If we consider $Re\, p(z) < \gamma + 1$ , $z \in U$ we have $Re\left[\dfrac{1}{(\gamma+1) - p(z)}\right] > 0$ and thus from Theorem 2.3 we obtain $p(z) \prec q(z)$ i.e.

$$\alpha < Re\left\{-\frac{zg'(z)}{g(z)}\right\} < \beta \, , \; z \in U.$$

Using the condition $Re\left\{-\dfrac{zg'(z)}{g(z)}\right\} < \gamma + 1$ we have $\alpha < Re\left\{-\dfrac{zg'(z)}{g(z)}\right\} < \beta'$, where $\beta' = min\{\beta, \gamma+1\}$ or $g \in \Sigma^*_{\alpha,\beta'}$ .

If we take $\beta > \gamma + 1$ from Theorem 3.2 we obtain:

**Corollary 3.1** *Let $0 \leq \alpha < 1$ , $\gamma > 0$ , $\beta > \gamma + 1$ and $G(z) \in \Sigma^*_{\alpha,\beta}$. If $g(z) = I_\gamma(G)(z)$, where $I_\gamma$ is defined in (2), and $g(z) \neq 0$ , $z \in \dot{U}$, then $g \in \Sigma^*_{\alpha,\gamma+1}$ .*

**Remark 3.3** *In the conditions from Corollary 3.1 with $\alpha = 0$ we have that $G(z) \in \Sigma^*_{0,\beta}$ and $g(z) = I_\gamma(G)(z)$ with $g(z) \neq 0$ , $z \in \dot{U}$ imply that $g(z) \in \Sigma^*_{0,\gamma+1}$.*

**Theorem 3.3** *Let $\gamma > 0$ , $\beta > \gamma + 1$ and $F(z) \in \Sigma^{cc}$ with respect to the function $G(z) \in \Sigma^*_{0,\beta}$ . If $f(z) = I_\gamma(F)(z)$, where $I_\gamma$ is defined in (2), with $f(z) \neq 0$ , $z \in \dot{U}$ and $g(z) = I_\gamma(G)(z)$ with $g(z) \neq 0$ , $z \in \dot{U}$, then $f(z) \in \Sigma^{cc}$ with respect to the function $g(z) \in \Sigma^*_{0,\gamma+1}$ .*

*Proof.*

In the given conditions it is sufficient to prove that

$$Re\left[-\frac{zf'(z)}{g(z)}\right] > 0 \, , \; z \in \dot{U}.$$

From (2) we have:

$$(\gamma+1)f(z) + zf'(z) = \gamma F(z) \tag{6}$$

and

$$(\gamma+1)g(z)+zg'(z)=\gamma G(z) \tag{7}$$

From (6) we have:

$$(\gamma+2)f'(z)+zf''(z)=\gamma F'(z) \tag{8}$$

Using (7) and (8) we have:

$$-\frac{zF'(z)}{G(z)}=-\frac{z\gamma F'(z)}{\gamma G(z)}=\frac{(\gamma+2)p(z)-\dfrac{z^2f''(z)}{g(z)}}{(\gamma+1)-h(z)} \tag{9}$$

where $p(z)=-\dfrac{zf'(z)}{g(z)}$ and $h(z)=-\dfrac{zg'(z)}{g(z)}$ with $p\in\mathcal{H}(U)$, $h\in\mathcal{H}(U)$ and $p(0)=h(0)=1$.

From $p(z)=-\dfrac{zf'(z)}{g(z)}$ we obtain:

$$-\frac{z^2f''(z)}{g(z)}=zp'(z)-p(z)\left[1+h(z)\right] \tag{10}$$

Using (10) we have from (9):

$$-\frac{zF'(z)}{G(z)}=p(z)+\frac{1}{(\gamma+1)-h(z)}\cdot zp'(z) \tag{11}$$

From $F(z)\in\Sigma^{cc}$ with respect to the function $G(z)\in\Sigma^*_{0,\beta}$ we have from (11):

$$p(z)+\frac{1}{(\gamma+1)-h(z)}\cdot zp'(z)\prec\frac{1+z}{1-z}$$

Using the Remark 3.3 we have from $G(z)\in\Sigma^*_{0,\beta}$ that $g(z)\in\Sigma^*_{0,\gamma+1}$ . Thus $0<Re\,h(z)<\gamma+1$. In this condition we have from Theorem 2.3 that $p(z)\prec\frac{1+z}{1-z}$ i.e. $Re\left[-\dfrac{zf'(z)}{g(z)}\right]>0$ , $z\in U$.

**Definition 3.4** *Let $\delta<1$ and $f(z)=\frac{1}{z}+\alpha_0+\alpha_1z+\cdots$ , $z\in U$ meromorphic and univalent in $\dot{U}$, with $f(z)\neq 0$ , $z\in\dot{U}$. We say that $f$ is close to convex of order $\delta$ with respect to the function $g(z)=\frac{1}{z}+\beta_0+\beta_1z+\cdots$ , $z\in U$ meromorphic and starlike of order $\delta$ in $U$, with $g(z)\neq 0$ , $z\in\dot{U}$, if $Re\left[-\dfrac{zf'(z)}{g(z)}\right]>\delta$ , $z\in\dot{U}$ . We denote this class with $\Sigma^{cc}(\delta)$ .*

**Remark 3.4** *We have* $\Sigma^{cc}(0)=\Sigma^{cc}$ *and* $\Sigma^{*}(\delta)\subset\Sigma^{cc}(\delta)$ .

**Theorem 3.4** *Let* $\gamma>0$ , $\beta>\gamma+1$ , $0\le\delta<1$ *and* $F(z)\in\Sigma^{cc}(\delta)$ *with respect to the function* $G(z)\in\Sigma^{*}_{\delta,\beta}$ . *If* $f(z)=I_\gamma(F)(z)$, *where* $I_\gamma$ *is defined in (2), with* $f(z)\neq 0$ , $z\in\dot{U}$ *and* $g(z)=I_\gamma(G)(z)$ *with* $g(z)\neq 0$ , $z\in\dot{U}$, *then* $f(z)\in\Sigma^{cc}(\delta)$ *with respect to the function* $g(z)\in\Sigma^{*}_{\delta,\gamma+1}$ .

*Proof.*

In a similarly way with the proof of the Theorem 3.3 we have:

$$-\frac{zF'(z)}{G(z)}=p(z)+\frac{1}{(\gamma+1)-h(z)}\cdot zp'(z)\prec q(z)$$

where $p(z)=-\dfrac{zf'(z)}{g(z)}$, $h(z)=-\dfrac{zg'(z)}{g(z)}$ and $q$ is convex with $q(U)=\Delta=\{w\in\mathbb{C}\ :\ Re\,w>\delta\}$ .

From Corollary 3.1 we have from $G(z)\in\Sigma^{*}_{\delta,\beta}$ that $g(z)\in\Sigma^{*}_{\delta,\gamma+1}$ and thus $Re\,h(z)<\gamma+1$ . In this condition from Theorem 2.3 we obtain $p(z)\prec q(z)$ i.e. $f(z)\in\Sigma^{cc}(\delta)$.

**Definition 3.5** *Let* $0\le\delta<1$ , $\beta>\delta$ *and* $f(z)=\frac{1}{z}+\alpha_0+\alpha_1 z+\cdots$ , $z\in U$ *meromorphic and univalent in* $\dot{U}$, *with* $f(z)\neq 0$ , $z\in\dot{U}$. *We say that* $f$ *is in the class* $\Sigma^{cc}_{\delta,\beta}$ *with respect to the function* $g(z)=\frac{1}{z}+\beta_0+\beta_1 z+\cdots$ , $z\in U$ *meromorphic in* $U$, *with* $g(z)\neq 0$ , $z\in\dot{U}$, *and* $g(z)\in\Sigma^{*}_{\delta,\beta}$, *if* $\delta<Re\left[-\dfrac{zf'(z)}{g(z)}\right]<\beta$ , $z\in\dot{U}$ .

**Remark 3.5** *We have* $\Sigma^{*}_{\delta,\beta}\subset\Sigma^{cc}_{\delta,\beta}$ *and if we consider* $\beta\to\infty$ *we have*$\Sigma^{cc}_{\delta,\infty}=\Sigma^{cc}(\delta)$ .

In a similarly way with the proof of the Theorem 3.4 we obtain:

**Theorem 3.5** *Let* $\gamma>0$ , $\beta>\gamma+1$ , $0\le\delta<1$ *and* $F(z)\in\Sigma^{cc}_{\delta,\beta}$ *with respect to the function* $G(z)\in\Sigma^{*}_{\delta,\beta}$ . *If* $f(z)=I_\gamma(F)(z)$, *where* $I_\gamma$ *is defined in (2), with* $f(z)\neq 0$ , $z\in\dot{U}$ *and* $g(z)=I_\gamma(G)(z)$ *with* $g(z)\neq 0$ , $z\in\dot{U}$, *then* $f(z)\in\Sigma^{cc}_{\delta,\beta}$ *with respect to the function* $g(z)\in\Sigma^{*}_{\delta,\gamma+1}$ .

## References

[1] S.S. Miller, P.T. Mocanu , *Differential subordinations and univalent functions* , Mich. Math. , 28(1981) , 157-171 .

[2] S.S. Miller, P.T. Mocanu , *Univalent solution of Briot-Bouquet differential equations* , J. Differential Equations , 56(1985) , 297-308 .

[3] S.S. Miller, P.T. Mocanu , *On some classes of first order differential subordinations* , Mich. Math. , 32(1985) , 185-195 .

[4] P.T. Mocanu, T. Bulboaca, G. Salagean , *Teoria geometrica a functiilor univalente* , Casa Cartii de Stiinta (Cluj) , 1999 .

[5] P.T. Mocanu, Gr. Salagean , *Integral operators and meromorphic starlike functions* , Mathematica (Cluj) , 32(55) , 2(1990) , 147-152 .

Address:

”Lucian Blaga” University
Department of Mathematics
550012 Sibiu
Romania

*E-mail*: acu_mugur@yahoo.com