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RIGIDITY OF TOROID FORMED BY REVOLUTION OF PARALLELOGRAM

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Abstract

One of the main tasks of the deformation theory is to point out to the rigid and flexible surfaces. In this paper we signify a torus like class of surfaces generated by parallelogram in E_3. It is proved that this class is rigid due to infinitesimal bending. Infinitesimal bending of generated surfaces is considered using Cohn-Vossen's method.

1 Introduction

The first result of the infinitesimal bending of the non-convex surface belongs to H. Liebman [6],[7]. He has proved that the torus and analytic surfaces containing the convex strip are rigid in a sense of infinitesimal bending.

In 1938 A. D. Alexandrov [1] has widened the above mentioned result of Liebman. He considered closed surfaces, divided in finite number of regions by piecewise smooth curves with constant Gaussian curvature. He called this surfaces T-surfaces and proved that analytical T-surfaces are rigid in a sense of the analytic infinitesimal bending.

Later, T. Minagawa and T. Rado enforced the results of H. Liebman [6], [7]. They have proved the rigidity of torus [9] and surface of revolution of class C^1, containing convex strip of class C^2 [11], on the presumption that the bending field is of the class C^1.

The results of H. Liebman on the rigidity of the torus and the ovaloid naturally led to the question of the existence of non-rigid closed surfaces. The first to answer this question was S. Cohn-Vossen [3], [5]. He has proved that from each plane curve we can get the meridian of non-rigid surface

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of revolution of genius 0. This result of S. Cohn-Vossen and his method, indicated many works on infinitesimal bending of non-convex surfaces of revolution.

Surfaces of revolution of genius 0 or 1 generated by rotation of broken (polygonal line) were considered by Cohn-Vossen, Bublik, K. M. Belov [2], N. G. Perlova [11].

Cohn-Vossen considered surfaces of genius 0 generated by polygonal line and concluded about the non-rigidity of some of them. K. M. Belov [2] has pointed out a class of toroids with meridian in shape of special quadrangle (with mutually perpendicular diagonals-one parallel to the axe of rotation), unlike parallelogram.

Generalization of the consideration from the paper [2] was given at [13]-[17] and [8]. Toroid surfaces non containing plane part, generated by triangular meridian are rigid [13].

We shall consider here infinitesimal bending of toroid rotational surfaces generated by meridian shaped like parallelogram. We also consider existence of the field of infinitesimal bending. The rigidity condition expressed by the coordinates of vertex of polygon, ie. by an analytical expression is considered geometrically.

2 The basic facts of the infinitesimal bending theory

We shall give the basic facts of the theory of infinitesimal bending of surfaces according to [4] and [5]. The basic concept used in this work can be defined in different ways.

2.1 Infinitesimal deformations of surfaces

Let's consider surface $S \subset \Re$ of the class $C^{\alpha}, \alpha \geq 3$.

Definition 2.1. *The surface S_ε is* **deformation** *of the surface S if it is included in continuous family of surfaces*

$$S_\varepsilon : \bar{r} = (u, v, \varepsilon) = \bar{r}_\varepsilon(u, v),\ (u, v) \in D \subset \Re^2,\ \varepsilon \in [0, 1],$$
$$\bar{r}_\varepsilon : D \times [0, 1] \to \Re^3,$$

and we get S for $\varepsilon = 0$.

We will here consider a kind of continuous family of surfaces, defining them according to [4].

Definition 2.2. *Let the surface*

$$S : \bar{r} = \bar{r}(u,v), \quad (u,v) \in D, \quad D \subset \Re^2 \tag{2.1}$$

be included in a family of surfaces

$$S_\varepsilon : \bar{r}_\varepsilon = \bar{r}_\varepsilon(u,v,\varepsilon), \quad (\varepsilon \geq 0, \varepsilon \to 0), \tag{2.2}$$

depending continuously on the parameter ε and we get S for $\varepsilon = 0$. In this way

$$S_\varepsilon : \bar{r}_\varepsilon = \bar{r}(u,v) + \varepsilon \overset{(1)}{\bar{z}}(u,v) + \varepsilon^2 \overset{(2)}{\bar{z}}(u,v) + ... + \varepsilon^m \overset{(m)}{\bar{z}}(u,v), \quad m \geq 1, \tag{2.3}$$

where $\overset{(j)}{\bar{z}}(u,v) \in C^\alpha (\alpha \geq 3), \quad j = 1, ..., m,$ are given fields, family S_ε is **infinitesimal deformation of the order m of the surface S.**

Theory considering geometric objects in connection with S_ε up to the precision of the order m with respect to $\varepsilon \quad (\varepsilon \to 0)$ is **infinitesimal deformation theory of surfaces of the order m.**

Giving different more special conditions we get different kinds of surface deformations.

Higher order deformations of polyhedral surfaces were considered at [12].

2.2 Infinitesimal bending of the first order

Let the regular surface S of the class $C^\alpha, \quad \alpha \geq 3$ be given in the vector form with (2.1) included in the family of surfaces

$$S_\varepsilon : \bar{r}_\varepsilon(u,v,\varepsilon) = \bar{r}(u,v) + \varepsilon \bar{z}(u,v), \tag{2.4}$$

where $\varepsilon (\varepsilon \to 0), (u,v) \in D, \ D \subset \Re$ and $\bar{r}_0(u,v,0) = \bar{r}(u,v)$.

Definition 2.3. *The surfaces (2.4) are* **infinitesimal bending of the first order** *of the surface S if*

$$ds_\varepsilon^2 - ds^2 = o(\varepsilon) \tag{2.5}$$

ie. if the difference of the squares of the line elements of this surfaces is of the order higher then the first.

The field $\bar{z}(u,v)$ for which

$$\frac{\partial \bar{r}(u,v,\varepsilon)}{\partial \varepsilon} = \bar{z}(u,v) \tag{2.6}$$

is **velocity** *or* **infinitesimal bending field** *of the infinitesimal bending.*

According to [4],[5] this definition is equivalent to the next theorem:

Theorem 2.1.*Necessary and sufficient condition for the surface S_ϵ (2.4) to be infinitesimal bending of the surface S (2.1) is*

$$d\bar{r}d\bar{z} = 0, \tag{2.7}$$

where $\bar{z}(u,v)$ is the velocity field at the initial moment of deformation.□

The equation (2.7) is equivalent to the next three partial differential equations:

$$\bar{r}_u\bar{z}_u = 0, \quad \bar{r}_u\bar{z}_v + \bar{r}_v\bar{z}_u = 0, \quad \bar{r}_v\bar{z}_v = 0. \tag{2.8}$$

Under infinitesimal bending of the surfaces each line element gets non-zero addition, which is the infinitesimal value of the second order with respect to ε, ie.

$$ds_\varepsilon - ds = o(\varepsilon) \geq 0. \tag{2.9}$$

Theorem 2.2.*Let $s = s(\varepsilon)$ be the arc length of the curve C_ε on the surface S_ε. Necessary and sufficient condition for the infinitesimal bending of the initial surface $S = S_0$ is*

$$\frac{\partial s_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} = 0, \tag{2.10}$$

i.e. the velocity of the change of arc length at the initial moment to be zero.□

Definition 2.4.*Bending field is* **trivial***, ie. it is a field of the rigid motion of the surface if it can be given in the form*

$$\bar{z} = \bar{a} \times \bar{r} + \bar{b}, \tag{2.11}$$

where $\bar{a}$ and $\bar{b}$ are constant vectors.

Definition 2.5.*A surface is* **rigid** *if it doesn't alow bending field other then trivial.*

If the surface is rigid of the order m it is rigid of the order n, $\quad n > m$.

3 Infinitesimal rigidity of toroid with meridian shaped as parallelogram

Theorem 3.1. *Toroid surface of revolution generated by a meridian in a shape of parallelogram, containing conical parts, is rigid.*

Proof. Let P_4 be the parallelogram with apices $A_i(u_i\rho_i), (i = 1, 2, 3, 4)$ considered at Descartes coordinate system $uO\rho$ with the axe of rotation u. The equations of the sides are:

$$
\begin{aligned}
&A_iA_{i+1} : \rho_{(i)} = \rho_i + \frac{\rho_{i+1} - \rho_i}{u_{i+1} - u_i}(u - u_i),\\
&\rho'_{(i)} = \frac{\rho_{i+1} - \rho_i}{u_{i+1} - u_i} = k_i \quad (i = 1, 2, 3, 4; A_5 \equiv A_1)
\end{aligned} \tag{3.1}
$$

where ρ_i is the value of ρ on A_iA_{i+1}.

In order to investigate infinitesimal bending of rotational surface we shall use Cohn-Vossen's method [5]. If we denote $\bar{e}$ is unit vector of the axis of rotation, $\bar{a}(v)$ unit vector of the ρ-axis, where v is the angle between the plane of initial position of the meridian and $\bar{a}(v)$ then $\bar{a}'(v)\perp\bar{a}(v)$ and $\bar{a}'(v)\perp\bar{e}$ (see [5], page 90, or [4] page 253). Radius vector of a surface of rotation, in the coordinate system with the base $\bar{e}, \bar{a}, \bar{a}'$ is

$$\bar{r}(u, v) = u\bar{e} + \rho(u)\bar{a}(v).$$

Fundamental field of infinitesimal bending of the surface we try to find in the form

$$
\begin{aligned}
&\bar{z}(u, v) = \bar{z}_k(u, v) = [\varphi_k(u)e^{ikv} + \tilde{\varphi}_k(u)e^{-ikv}]\bar{e}+\\
&[\psi_k(u)e^{ikv} + \tilde{\psi}_k(u)e^{-ikv}]\bar{a}(v) + [\chi_k(u)e^{ikv} + \tilde{\chi}_k(u)e^{-ikv}]\bar{a}'(v),
\end{aligned}
$$

where $\tilde{\varphi}_k(u)$ conjugated complex value for $\varphi_k(u)$. The functions $\varphi_k(u)$, $\psi_k(u)$, $\chi_k(u)$ satisfy the equations

$$
\begin{aligned}
\varphi'_k(u) + \rho'(u)\psi'_k(u) &= 0,\\
\psi_k(u) + ik\chi'_k(u) &= 0,\\
ik\psi_k(u) + \rho'(u)[ik\psi_k(u) - \chi_k(u)] + \rho(u)\chi'_k(u) &= 0.
\end{aligned}
$$

Functions $\psi_k(u)$, $\chi_k(u)$ satisfy also the equation

$$\rho(u)\lambda''(u) + (k^2 - 1)\rho''(u)\lambda(u) = 0, \tag{3.2}$$

where $\lambda(u)$ is unknown function. We omit index k, and denote with $\psi_{(i)}$ the value of the function ψ on A_iA_{i+1}, $\quad i = 1, 2, 3, 4, \quad A_5 \equiv A_1$.

¿From the equations (3.1) and (3.2) it follows also the linearity of the functions $\psi_i(u)$

$$\psi_{(i)} = M_i u + N_i \quad (i = 1, 2, 3, 4) \tag{3.3}$$

At the points $u = \sigma$ of the meridian, where $\rho(\sigma - 0) = \rho(\sigma + 0)$, ie. at the apices of the parallelogram, continuity of the function $\psi_{(i)}(u)$ gives us

$$\psi_{(i)}(u_i) = \psi_{(i-1)}(u_i), \quad i = 1, 2, 3, 4; \quad \psi_{(0)}(u_1) = \psi_{(4)}(u_1)$$

and from there based on (3.3)

$$M_i u_i + N_i = M_{i-1} u_i + N_{i-1} \quad (i = 1, 2, 3, 4); \quad M_0 \equiv M_4, \quad N_0 \equiv N_4$$

Considering this system as a system with respect to unknowns N_i, $i = 1, 2, 3, 4$, we get

$$\begin{aligned} N_1 \qquad\qquad\qquad - N_4 &= -M_1 u_1 + M_4 u_1 \\ N_1 - N_2 \qquad\qquad\qquad &= -M_1 u_2 + M_2 u_2 \\ N_2 - N_3 \qquad\quad &= -M_2 u_3 + M_3 u_3 \\ N_3 - N_4 &= -M_3 u_4 + M_4 u_4 \end{aligned} \tag{3.4}$$

At the apices of the polygon according to Cohn-Vossen we have the next equation

$$\rho(\sigma)[\psi_k'(\sigma + 0) - \psi_k'(\sigma - 0)] + (k^2 - 1)\psi_k(\sigma)[\rho'(\sigma + 0) - \rho'(\sigma - 0)] = 0,$$

applying this equation to the apices M_i, $\quad i = 1, 2, 3, 4$ we get the system of the equations

$$\begin{gathered} \rho_i(M_i - M_{i-1}) + (k^2 - 1)((M_i u_i + N_i)(k_i - k_{i-1}) = 0, \\ (i = 1, 2, 3, 4; \quad M_0 \equiv M_4, \quad k_0 \equiv k_4) \end{gathered} \tag{3.5}$$

The equations (3.4) and (3.5) present the system of linear equations with respect to unknowns M_i,N_i, $\quad (i = 1, 2, 3, 4)$. Let us consider the system (3.4) with respect to N_i. We shall do the elementary transformations on rows of the matrix of the system. This steps transform system into next one with unknowns N_1, N_2, N_3, N_4

$$\begin{aligned} N_1 \qquad\qquad\qquad -N_4 &= (M_4 - M_1)u_1 \\ -N_2 \qquad\qquad +N_4 &= (M_1 - M_4)u_1 + (M_2 - M_1)u_2 \\ - N_3 \quad +N_4 &= (M_1 - M_4)u_1 + \sum_{l=2}^{3}(M_l - M_{l-1})u_l \\ 0 &= (M_1 - M_4)u_1 + \sum_{l=2}^{4}(M_l - M_{l-1})u_l(M_l - M_{l-1})u_l \end{aligned}$$

The system is compatible if and only if rank of matrix of the system is equal to the rank of the extended matrix of this system, that lead us to

$$(M_1 - M_4)u_1 + \sum_{l=2}^{4}(M_l - M_{l-1})u_l(M_l - M_{l-1})u_l = 0$$

ie.

$$M_4 = \frac{1}{u_1 - u_4}\sum_{i=1}^{n-1}(u_i - u_{i+1})M_i.$$

Further we can express N_1, N_2, N_3 in (3.4) over N_4, M_1, M_2, M_3 and replace them in (3.5). This way gives us system of algebraic homogenous linear equations with determinant of the coefficients A. Necessary and sufficient condition to have nontrivial solution is

$$detA = 0.$$

Condition for the existence of the field of infinitesimal bending is

$$\begin{aligned}
&[\rho_1\rho_2u_{43}k_{32} + \rho_1\rho_3u_{24}k_{12} + \rho_2\rho_3u_{14}k_{41}\\
&\ + \rho_1(k^2-1)u_{23}u_{43}k_{12}k_{23}]\times\\
&[\rho_4u_{12}u_{31}k_{41} + (k^2-1)u_{12}u_{43}u_{14}k_{14}k_{34} + \rho_1u_{43}u_{24}k_{34}]-\\
&-(\rho_1u_{23}u_{43}k_{23} + \rho_3u_{12}u_{14}k_{41})\\
&\ \times[\rho_1\rho_2u_{34}k_{34} + \rho_1\rho_4u_{32}k_{12} + \rho_2\rho_4u_{31}k_{41}\\
&\ + \rho_2(k^2-1)u_{14}u_{43}k_{14}k_{34}] = 0,
\end{aligned} \tag{3.6}$$

where

$$u_i - u_j = u_{ij}$$
$$k_i - k_j = k_{ij}.$$

On the Fig.1. is shown an example of toroid obtained by revolution of meridian shaped as parallelogram around u axe. Fig.1. is obtained using symbolic program package *Mathematica*.

We shall examine four possible cases of parallelogram non containing plane parts generated by sides perpendicular to the axe of rotation, and cylindrical parts generated by sides parallel to the axe of rotation. Following figures show four different cases of parallelograms and examine rigidity conditions on them. Parallelogram's vertices are expressed in terms of positive values a, b, c, d and $e > 0$.

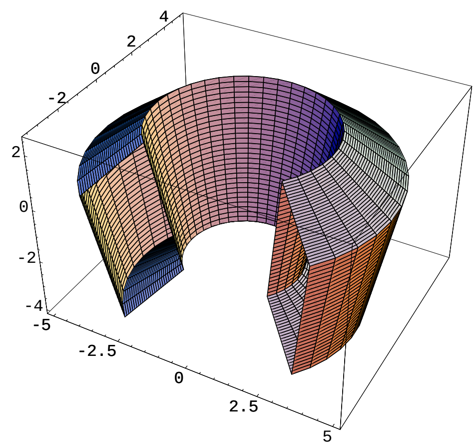

Fig.1.

Case 1:

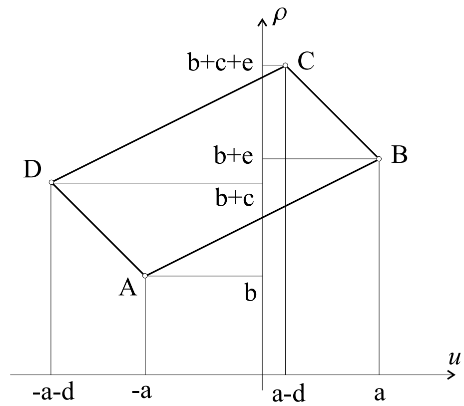

Fig.2.

On the Fig.2. is shown parallelogram with apices $A[-a, b]$, $B[a, b+e]$, $C[a-d, b+c+e]$ and $D[-a-d, b+c]$. In this case non rigidity conditions given by (3.6) is

$$-\frac{b(2ac+ed)^4k^4}{4a^2d} = 0.$$

It is easy to see that expression on the left can not be equal 0 as it is always less then 0 according to assumption a, b, c, d and e greater then 0. This means that field of infinitesimal bending for this case does not exist.

Case 2:

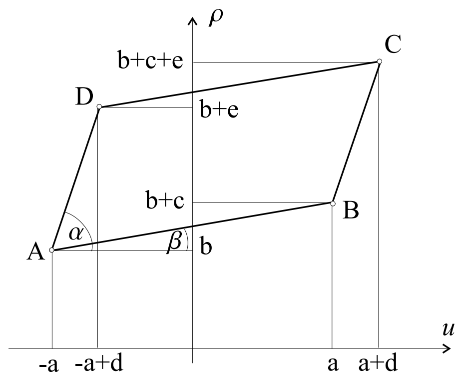

Fig.3.

On the Fig.3. is shown parallelogram with apices $A[-a, b]$, $B[a, b+c]$, $C[a+d, b+c+e]$ and $D[-a+d, b+e]$. In this case non rigidity conditions given by (3.6) is

$$-\frac{b(-2ac+ed)^4k^4}{4a^2d} = 0.$$

Expression on the left can be equal 0 if and only if

$$-2ac+ed = 0.$$

This condition can be written as

$$\frac{e}{d} = \frac{c}{2a} \Leftrightarrow tg(\alpha) = tg(\beta).$$

If this is true, then parallelogram does not exist, as its vertices lie on a line.

Case 3:

On the Fig.4. is shown parallelogram with apices $A[-a, b+c]$, $B[a, b]$, $C[a-d, b+e]$ and $D[-a-d, b+c+e]$. In this case non rigidity conditions given by (3.6) is

$$\frac{-(b+c)(-2ae+cd)^4k^4}{4a^2d} = 0.$$

Expression on the left can be 0 if and only if

$$-2ae+cd = 0.$$

This condition can be written as

$$\frac{c}{2a} = \frac{e}{d} \Leftrightarrow tg(\delta) = tg(\gamma)$$

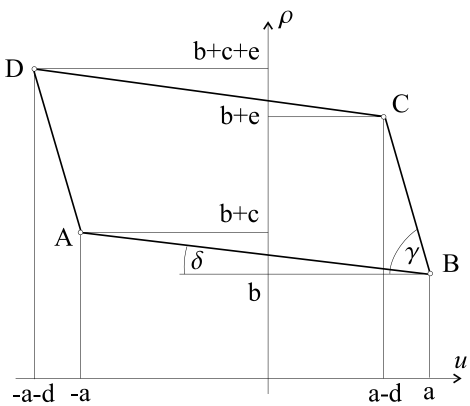

Fig.4.

If this is true, vertices of parallelogram lie on a line and than parallelogram does not exist.

Case 4:

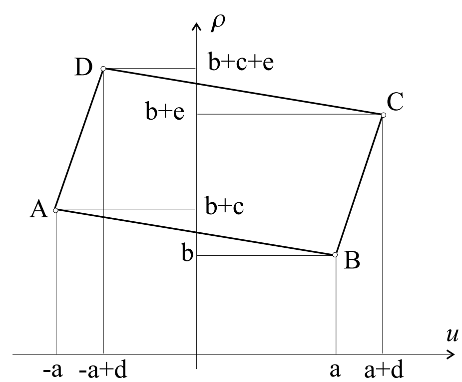

Fig.5.

On the Fig.5. is shown parallelogram with apices $A[-a, b+c]$, $B[a, b]$, $C[a+d, b+e]$ and $D[-a+d, b+c+e]$. In this case non rigidity conditions given by (3.6) is

$$-\frac{(b+c)(2ae+cd)^4k^4}{4a^2d} = 0.$$

According to assumption it is easy to see that expression on the left is never $= 0$, actually it is always < 0, so field of infinitesimal bending does not exist for this case also.□

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