Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **21:2** (2007), 121–131

ON SOME CLOSE TO CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

Mugur Acu and Irina Dorca

Abstract

In this paper we propose for study a class of close to convex functions with negative coefficients defined by using a modified Sălăgean operator.

1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U,

$$A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$$

and $S = \{f \in A : f \text{ is univalent in } U\}$.

In [7] the subfamily T of S consisting of functions f of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ \ a_j \geq 0, j = 2, 3, ...,\ \ z \in U \tag{1}$$

was introduced.

The purpose of this paper is to define a class of close to convex functions with negative coefficients and to give some properties of its by using a modified Sălăgean operator.

2000 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Close to convex functions with negative coefficients, integral operators, generalized Salagean operator.
Received: May 8, 2007

2 Preliminary results

Let D^n be the Sălăgean differential operator (see [5]) $D^n : A \to A$, $n \in \mathbb{N}$, defined as:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z))$$

Remark 2.1 *If* $f \in T, f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j,\ a_j \geq 0,\ j = 2, 3, ...,\ z \in U$ *then* $D^n f(z) = z - \sum\limits_{j=2}^{\infty} j^n a_j z^j$.

Definition 2.1 *[2] Let* $\beta, \lambda \in \mathbb{R},\ \beta \geq 0,\ \lambda \geq 0$ *and* $f(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$. *We denote by* D_λ^β *the linear operator defined by*

$$D_\lambda^\beta : A \to A\,,$$

$$D_\lambda^\beta f(z) = z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^\beta\, a_j z^j\,.$$

Theorem 2.1 *[6] If* $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j,\ a_j \geq 0,\ j = 2, 3, ...,\ z \in U$ *then the next assertions are equivalent:*

(i) $\sum\limits_{j=2}^{\infty} j a_j \leq 1$

(ii) $f \in T$

(iii) $f \in T^*$, *where* $T^* = T \bigcap S^*$ *and* S^* *is the well-known class of starlike functions.*

Definition 2.2 *[6] Let* $\alpha \in [0, 1)$ *and* $n \in \mathbb{N}$, *then*

$$S_n(\alpha) = \left\{ f \in A : Re \frac{D^{n+1} f(z)}{D^n f(z)} > \alpha, z \in U \right\}$$

is the set of n-starlike functions of order α.
Also, we denote $T_n(\alpha) = T \bigcap S_n(\alpha)$.

Definition 2.3 *[5] Let $I_c : A \to A$ be the integral operator defined by $f = I_c(F)$, where $c \in (-1, \infty)$, $F \in A$ and*

$$f(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} F(t) dt. \tag{2}$$

We note if $F \in A$ is a function of the form (1), then

$$f(z) = I_c F(z) = z - \sum_{j=2}^{\infty} \frac{c+1}{c+j} a_j z^j. \tag{3}$$

We denote by $f * g$ the modified Hadamard product of two functions $f(z)$, $g(z) \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j, (a_j \geq 0,\ j = 2, 3, ...)$ and $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$, ($b_j \geq 0$, j=2,3,...), is defined by

$$(f * g)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j.$$

Definition 2.4 *[3] Let $f \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ...$, $z \in U$. We say that f is in the class $TL_\beta(\alpha)$ if:*

$$Re \frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} > \alpha, \quad \alpha \in [0, 1), \quad \lambda \geq 0, \quad \beta \geq 0, \quad z \in U.$$

Theorem 2.2 *[3] Let $\alpha \in [0, 1)$, $\lambda \geq 0$ and $\beta \geq 0$. The function $f \in T$ of the form (1) is in the class $TL_\beta(\alpha)$ iff*

$$\sum_{j=2}^{\infty} [(1 + (j-1)\lambda)^\beta (1 + (j-1)\lambda - \alpha)] a_j < 1 - \alpha. \tag{4}$$

Theorem 2.3 *[3] If $f(z) = z - \sum_{j=2}^{\infty} a_j z^j \in TL_\beta(\alpha), (a_j \geq 0,\ j = 2, 3, ...)$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j \in TL_\beta(\alpha)$, $(b_j \geq 0,\ j = 2, 3, ...)$, $\alpha \in [0, 1)$, $\lambda \geq 0$, $\beta \geq 0$, then $f(z) * g(z) \in TL_\beta(\alpha)$.*

Theorem 2.4 *[3] If* $F(z) = z - \sum_{j=2}^{\infty} a_j z^j \in TL_\beta(\alpha)$*, then* $f(z) = I_c F(z) \in TL_\beta(\alpha)$*, where* I_c *is the integral operator defined by (2).*

In [3] it's considered the integral operator $I_{c+\delta} : A \to A$, $0 < u \le 1$, $1 \le \delta < \infty$, $0 < c < \infty$, defined by

$$f(z) = I_{c+\delta}(F(z)) = (c+\delta) \int_0^1 u^{c+\delta-2} F(uz) du. \tag{5}$$

Remark 2.2 *For* $F(z) = z + \sum_{j=2}^{\infty} a_j z^j$*. From (5) we obtain*

$$f(z) = z + \sum_{j=2}^{\infty} \frac{c+\delta}{c+j+\delta-1} a_j z^j.$$

Also, we notice that $0 < \dfrac{c+\delta}{c+j+\delta-1} < 1$, where $0 < c < \infty$, $j \ge 2$, $1 \le \delta < \infty$.

Remark 2.3 *It is easy to prove that for* $F(z) \in T$ *and* $f(z) = I_{c+\delta}(F(z))$*, we have* $f(z) \in T$*, where* $I_{c+\delta}$ *is the integral operator defined by (5).*

Theorem 2.5 *[3] Let* $F(z)$ *be in the class* $TL_\beta(\alpha)$*,* $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$*,* $a_j \ge 0$*,* $j \ge 2$*. Then* $f(z) = I_{c+\delta}(F(z)) \in TL_\beta(\alpha)$*, where* $I_{c+\delta}$ *is the integral operator defined by (5) .*

3 Main results

Definition 3.1 *Let* $f \in T$*,* $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$*,* $a_j \ge 0$*,* $j = 2, 3, ...$*,* $z \in U$ *and* $g(z) \in TL_\beta(\alpha)$*. We say that* f *is in the class* $CCTL_\beta(\alpha)$ *if:*

$$Re \frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta g(z)} > \alpha, \quad \alpha \in [0,1), \quad \lambda \ge 0, \quad \beta \ge 0, \quad z \in U.$$

Theorem 3.1 *Let $\alpha \in [0,1)$, $\lambda \geq 0$ and $\beta \geq 0$. The function $f \in CCTL_\beta(\alpha)$ with respect to the function $g(z) \in TL_\beta(\alpha)$ iff*

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta[(1+(j-1)\lambda)a_j+(2-\alpha)b_j] < 1-\alpha\,. \tag{6}$$

Proof. Let $f \in CCTL_\beta(\alpha)$, $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $a_j \geq 0, j \geq 2$, with respect to the function $g(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j \in TL_\beta(\alpha)$, $b_j \geq 0, j \geq 2$, where $\alpha \in [0,1)$, $\lambda \geq 0$ and $\beta \geq 0$. We have

$$Re\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^\beta g(z)} > \alpha\,.$$

If we take $z \in [0,1)$, $\beta \geq 0$, $\lambda \geq 0$, we have (see Definition 2.1):

$$\frac{1-\sum\limits_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}a_j z^{j-1}}{1-\sum\limits_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}b_j z^{j-1}} > \alpha\,. \tag{7}$$

From $g(z) \in TL_\beta(\alpha)$ we notice that $1-\sum\limits_{j=2}^{\infty}[1+(j-1)\lambda]^\beta b_j z^{j-1} > 0$. We obtain

$$1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}a_j z^{j-1} > \alpha-\alpha\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}b_j z^{j-1},$$

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta[(1+(j-1)\lambda)a_j-\alpha b_j z^{j-1}] < 1-\alpha.$$

Letting $z \to 1^-$ along the real axis we have:

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta[(1+(j-1)\lambda)a_j-\alpha b_j] < 1-\alpha. \tag{8}$$

From

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta[(1+(j-1)\lambda)a_j-\alpha b_j] \leq \sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta[(1+(j-1)\lambda)a_j+(2-\alpha)b_j]$$

we notice that if the following condition hold

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(1+(j-1)\lambda)a_j+(2-\alpha)b_j]<1-\alpha\,,$$

then also the condition (8) holds.

Conversely, let take $f \in T$ with respect to the function $g(z) \in TL_{\beta}(\alpha)$, for which the condition (6) hold.

The condition $Re\dfrac{D_{\lambda}^{\beta+1}f(z)}{D_{\lambda}^{\beta}g(z)} > \alpha$ is equivalent with

$$\alpha - Re\left(\frac{D_{\lambda}^{\beta+1}f(z)}{D_{\lambda}^{\beta}g(z)}-1\right)<1\,. \tag{9}$$

We have

$$\alpha - Re\left(\frac{D_{\lambda}^{\beta+1}f(z)}{D_{\lambda}^{\beta}g(z)}-1\right) \le \alpha + \left|\frac{D_{\lambda}^{\beta+1}f(z)}{D_{\lambda}^{\beta}g(z)}-1\right|$$

$$= \alpha + \left|\frac{D_{\lambda}^{\beta+1}f(z)-D_{\lambda}^{\beta}g(z)}{D_{\lambda}^{\beta}g(z)}\right|$$

$$= \alpha + \left|\frac{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(1+(j-1)\lambda)a_j-b_j]z^{j-1}}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}b_jz^{j-1}}\right|$$

$$\le \alpha + \frac{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}|(1+(j-1)\lambda)a_j-b_j|\cdot|z|^{j-1}}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}b_j|z|^{j-1}}$$

$$< \frac{\alpha+\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[|(1+(j-1)\lambda)a_j-b_j|-\alpha b_j]}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}b_j}$$

$$\leq \frac{\alpha + \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(1+(j-1)\lambda)a_j + b_j - \alpha b_j]}{1 - \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta} b_j}$$

$$= \frac{\alpha + \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(1+(j-1)\lambda)a_j + (1-\alpha) b_j]}{1 - \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta} b_j} < 1\,.$$

Thus

$$\alpha + \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(1+(j-1)\lambda)a_j + (1-\alpha)b_j] < 1 - \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta} b_j\,,$$

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(1+(j-1)\lambda)a_j + (2-\alpha)b_j] < 1-\alpha$$

which is the condition (6).

Remark 3.1 *If we take $f \equiv g$ we obtain from Theorem 3.1*

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(1+(j-1)\lambda) + (2-\alpha)]a_j < 1-\alpha\,,$$

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[3+(j-1)\lambda-\alpha]a_j < 1-\alpha\,.$$

From

$$1-\alpha > \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[3+(j-1)\lambda-\alpha]a_j > \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[1+(j-1)\lambda-\alpha]a_j$$

we obtain

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[1+(j-1)\lambda-\alpha]a_j < 1-\alpha$$

which is the condition from Theorem 2.2.

Remark 3.2 *From the proof of Theorem 3.1 we obtain a necessary condition for a function* $f \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, *to be in the class* $CCTL_\beta(\alpha)$ *with respect to the function* $g(z) \in TL_n(\alpha)$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$, $a_j \geq 0$, $b_j \geq 0, j \geq 2, n \in \mathbb{N}, \alpha \in [0,1)$, *which is*

$$\sum_{j=2}^{\infty} [1 + (j-1)\lambda]^{\beta} [(1 + (j-1)\lambda) a_j - \alpha b_j] < 1 - \alpha \,.$$

Remark 3.3 *Using the condition (6) it is easy to prove that* $CCTL_{\beta+1}(\alpha) \subseteq CCTL_\beta(\alpha)$, *where* $\beta \geq 0$, $\alpha \in [0,1)$ *and* $\lambda \geq 0$.

Theorem 3.2 *If* $f^1(z) = z - \sum_{j=2}^{\infty} a_j^1 z^j \in CCTL_\beta(\alpha), (a_j^1 \geq 0,\ j = 2,3,...)$, *with respect to the function* $g^1(z) = z - \sum_{j=2}^{\infty} b_j^1 z^j \in TL_\beta(\alpha)$, $(b_j^1 \geq 0,\ j = 2,3,...)$ *and* $f^2(z) = z - \sum_{j=2}^{\infty} a_j^2 z^j \in CCTL_\beta(\alpha), (a_j^2 \geq 0,\ j = 2,3,...)$, *with respect to* $g^2(z) = z - \sum_{j=2}^{\infty} b_j^2 z^j \in TL_\beta(\alpha)$, $(b_j^2 \geq 0,\ j = 2,3,...)$, $\alpha \in [0,1)$, $\lambda \geq 0$, $\beta \geq 0$, *then* $f^1(z) * f^2(z) \in CCTL_\beta(\alpha)$ *with respect to the function* $g^1(z) * g^2(z)$.

Proof. From Theorem 2.3 we have $g^1(z) * g^2(z) \in TL_\beta(\alpha)$. We have

$$\sum_{j=2}^{\infty} [1 + (j-1)\lambda]^{\beta} [(1 + (j-1)\lambda) a_j^1 + (2 - \alpha) b_j^1] < 1 - \alpha$$

and

$$\sum_{j=2}^{\infty} [1 + (j-1)\lambda]^{\beta} [(1 + (j-1)\lambda) a_j^2 + (2 - \alpha) b_j^2] < 1 - \alpha \,.$$

We have from Definition 2.5 that $f^1(z) * f^2(z) = z - \sum_{j=2}^{\infty} a_j^1 a_j^2 z^j$ and $g^1(z) * g^2(z) = z - \sum_{j=2}^{\infty} b_j^1 b_j^2 z^j$. From Theorem 2.1 when $f^2(z) \in T$ we have $\sum_{j=2}^{\infty} j a_j^2 \leq 1$. So, $a_j^2 < 1, j = 2,3,...$. Similarly we obtain $b_j^2 < 1, j = 2,3,...$.

Thus (using $\sum\limits_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(1+(j-1)\lambda)a_j^1+(2-\alpha)b_j^1]<1-\alpha$):

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(1+(j-1)\lambda)a_j^1a_j^2+(2-\alpha)b_j^1b_j^2]$$

$$<\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(1+(j-1)\lambda)a_j^1+(2-\alpha)b_j^1]<1-\alpha\,.$$

This means that $f^1(z)*f^2(z)\in CCTL_{\beta}(\alpha)$ with respect to the function $g^1(z)*g^2(z)\in TL_{\beta}(\alpha)$.

Theorem 3.3 *If* $F(z)=z-\sum\limits_{j=2}^{\infty}a_j^1z^j\in CCTL_{\beta}(\alpha)$ *with respect to the function* $G(z)=z-\sum\limits_{j=2}^{\infty}b_j^1z^j\in TL_{\beta}(\alpha)$, *then* $f(z)=I_cF(z)\in CCTL_{\beta}(\alpha)$ *with respect to* $g(z)=I_cG(z)\in TL_{\beta}(\alpha)$, *where* I_c *is the integral operator defined by (2).*

Proof. From Theorem 2.4 we have $g(z)=I_cG(z)\in TL_{\beta}(\alpha)$ and $g(z)=z-\sum\limits_{j=2}^{\infty}b_j^2z^j$, where $b_j^2=\dfrac{c+1}{c+j}b_j^1$, $c\in(-1,\infty)$, $j\geq 2$.

We have $f(z)=z-\sum\limits_{j=2}^{\infty}a_j^2z^j$, where $a_j^2=\dfrac{c+1}{c+j}a_j^1$, $c\in(-1,\infty)$, $j\geq 2$.

Thus $a_j^2<a_j^1$ and $b_j^2<b_j^1$ and using the condition (6) for $F(z), z\in U$, we obtain

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(1+(j-1)\lambda)a_j^2+(2-\alpha)b_j^2]$$

$$<\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(1+(j-1)\lambda)a_j^1+(2-\alpha)b_j^1]<1-\alpha\,.$$

This completes our proof.

Theorem 3.4 *Let* $F(z)$ *to be in the class* $CCTL_{\beta}(\alpha)$, $F(z)=z-\sum\limits_{j=2}^{\infty}a_jz^j$, *with respect to the function* $G(z)=z-\sum\limits_{j=2}^{\infty}b_jz^j\in TL_{\beta}(\alpha)$, $a_j,b_j\geq 0$, $\beta,\lambda\geq 0, j\geq 2,\alpha\in[0,1)$. *Then* $f(z)=I_{c+\delta}(F(z))\in CCTL_{\beta}(\alpha)$ *with respect to* $g(z)=I_{c+\delta}(G(z))\in TL_{\beta}(\alpha)$, *where* $I_{c+\delta}$ *is the integral operator defined by (4).*

Proof. From Theorem 2.5 we have $g(z) = I_{c+\delta}(G(z)) \in TL_\beta(\alpha)$, where $g(z) = z - \sum_{j=2}^{\infty} \frac{c+\delta}{c+\delta+j-1} b_j$, $0 < \frac{c+\delta}{c+\delta+j-1} < 1$, where $0 < c < \infty$, $j \geq 2$.

From $F(z) \in CCTL_\beta(\alpha)$ we have (see Theorem 3.1)

$$\sum_{j=2}^{\infty} [1+(j-1)\lambda]^\beta [(1+(j-1)\lambda)a_j + (2-\alpha)b_j] < 1-\alpha\,,$$

where $a_j, b_j, \beta, \lambda \geq 0, j \geq 2, \alpha \in [0,1)$. From Remark 2.2 we obtain $f(z) = z - \sum_{j=2}^{\infty} \frac{c+\delta}{c+\delta+j-1} a_j$ with $0 < \frac{c+\delta}{c+\delta+j-1} < 1$, where $0 < c < \infty$, $j \geq 2$.

Thus (using $\sum_{j=2}^{\infty} [1+(j-1)\lambda]^\beta [(1+(j-1)\lambda)a_j + (2-\alpha)b_j] < 1-\alpha$)

$$\sum_{j=2}^{\infty} [1+(j-1)\lambda]^\beta [(1+(j-1)\lambda) \frac{c+\delta}{c+\delta+j-1} a_j + (2-\alpha) \frac{c+\delta}{c+\delta+j-1} b_j]$$

$$< \sum_{j=2}^{\infty} [1+(j-1)\lambda]^\beta [(1+(j-1)\lambda)a_j + (2-\alpha)b_j] < 1-\alpha\,.$$

This completes our proof.

Remark 3.4 *We notice that in the particulary case, obtained for* $\lambda = 1$ *and* $\beta \in \mathbb{N}$, *we find similarly results for the class* $CCT_n(\alpha)$ *of the n-close to convex functions of order* α *with negative coefficients (see [1]).*

References

[1] M. Acu, *On a subclass of functions with negative coefficients*, General Mathematics, Vol. 10, No. 3-4 (2002), 57-66.

[2] M. Acu, S. Owa, *Note on a class of starlike functions*, Proceeding Of the International Short Joint Work on Study on Calculus Operators in Univalent Function Theory - Kyoto 2006, 1-10.

[3] M. Acu, I. Dorca, S. Owa, *On some starlike functions with negative coefficients* (to appear).

[4] D. Blezu, *On the n-uniform close to convex functions with respect to a convex domain*, General Mathematics, Vol. 9, Nr. 3 - 4, 3 - 14.

[5] G. S. Sălăgean, *On some classes of univalent functions*, Seminar of geometric function theory, Cluj - Napoca, 1983.

[6] G. S. Sălăgean, *Geometria Planului Complex*, Ed. Promedia Plus, Cluj - Napoca, 1999.

[7] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. 5(1975), 109-116.

Address:

University "Lucian Blaga" of Sibiu, Department of Mathematics, Str. Dr. I. Raţiu, Nr. 5–7, 550012 - Sibiu, Romania.

E-mail

Mugur Acu: acu_mugur@yahoo.com

Irina Dorca: ira_dorca@yahoo.com