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ESTIMATION IN REAL DATA SET BY SPLIT–ARCH MODEL

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Abstract

Famous models of conditional heteroscedasticity describe various effects of behavior of the financial markets. In this paper, we investigate the related model, called Split–ARCH, in some of its stochastic aspects, as the necessary and sufficient conditions of the strong stationarity and the estimation procedure. The basic asymptotic properties of those estimates are described, too. The most important segment of our work is dedicated to the practical issue of Split–ARCH model in analysis of the dynamics of the real data. We compared the Split–ARCH with standard models of ARCH type and showed that it was better stochastic model for the explanation of the world market prices of some precious metals.

1 Introduction

The stochastic analysis of financial sequences is commonly based on the time series modelling of data set which will be able to describe the distribution or behavior of a real data. It has been shown empirically that the most of financial series exhibit nonlinear changes in the dynamics which will obviously imply nonlinearity of the corresponding stochastic models. The starting point of these models is the market price flow usually denoted as

$$S = (S_n)_{n \in D}$$

where $D \subseteq \mathbb{Z}$. It is considered to be the sequence of random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$ expanded by the filtration

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$\mathcal{F} = (\mathcal{F}_n)$ of nondecreasing σ–algebras on Ω. Applying the efficient market's continuous compounding principle, the price S_n can be represented as

$$S_n = S_0 e^{H_n}, \tag{1.1}$$

where the compound return $H_n = \sum_{k=1}^{n} h_k$ is the sum of $\mathcal{F}_n$ adaptive daily returns h_n. So, from the statistical point of view, the main problem is to detect the probability behavior of the sequence $h = (h_n)_{n \in D}$. Conditional heteroscedastic models are based on the following definition of the daily returns

$$h_n = \sigma_n \varepsilon_n, \quad n \in D \tag{1.2}$$

where (σ_n) is the sequence of $\mathcal{F}_{n-1}$ adaptive random variables well known as the volatility, and (ε_n) is the white noise sequence of (0,1) i.i.d. $\mathcal{F}_n$ adaptive random variables. So, h_n has, for all $n \in D$, the appropriate expectation as follows

$$E(h_n) = E\left[E(h_n|\mathcal{F}_{n-1})\right] = 0, \quad Var(h_n) = E\left[E(h_n^2|\mathcal{F}_{n-1})\right] = E(\sigma_n^2)$$

and it can be easily seen that h_n is a martingale difference for all n, i. e. $h = (h_n)$ is the sequence of uncorrelated random variables (we illustrate some typical empirical situations in Figure 1).

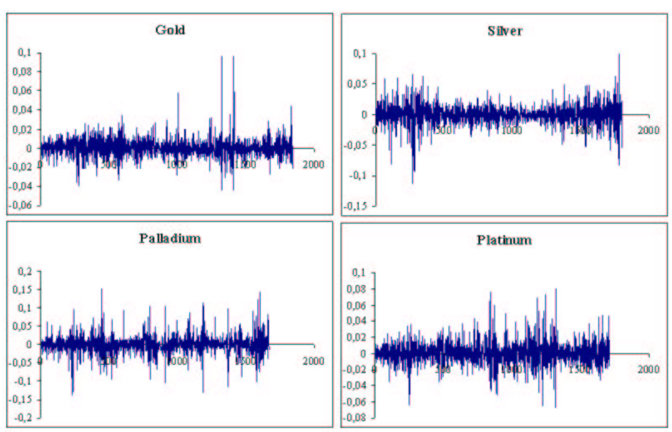

Figure 1: The log-returns of the fluctuations of the precious metals prices in the period 1998–2005. (Source: Kitco Inc. & London Fix database)

The class of the stochastic processes called *autoregressive conditional heteroscedastic* (*ARCH*) *process* was introduced by Engle [6] and successfully applied in many of the analysis of financial markets. Tim Bolerslev [2] spread this idea when he introduced *generalized autoregressive conditional heteroscedastic* (*GARCH*) *model*. These two models were able to explain a number of the properties of financial indexes (first of all heavy tails and clustering).

But, many of the empirical data sets indicate outstanding nonlinearity of the empirical volatility $\hat{\sigma}_n$ which can be manifested in the various manners, as the sharp growth of the price occurs in a relatively short time intervals, and, consequently, empirical volatility grows extremely sharp. In order to solve these problems, many generalizations of standard ARCH and GARCH models were done, for example, Zakoian [15] or Fornari and Mele [7]. In [12] we introduce the new model of the conditional heteroscedasticity that had been named *Split–ARCH model*. It explained, subject to the experimental investigation, the 'small sample' experiment – soybean meal price data from Product Exchange Novi Sad and a 'large sample' one – oil price data according to the Wall Street Journal source. In this way, our model describes nonlinear behavior of volatility caused by the great fluctuation of price which will properly correspond to the changes of values of (h_n) and, also, the volatility sequence (σ_n^2). The general definition of Split–ARCH will follow the equation (1.2) and the following one

$$\sigma_n^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i h_{n-i}^2 + \sum_{j=1}^{q} f_j(\sigma_{n-j}^2) I\left(\varepsilon_{n-j}^2 \geq c\right), \quad n \in D\,. \tag{1.3}$$

The coefficients of the model satisfy the conditions $\alpha_0 > 0$ and $\alpha_i \geq 0$, $i = 1, 2, \ldots, p$, while $f_j = f_j(u)$, $j = 1, \ldots, q$, $u \geq 0$ is a nonnegative $\mathcal{F}_{n-j}$ measurable function of the volatility sequence which will specify the reaction to the extremely large values in (ε_n). Obviously, it will be difficult to discuss the model and its properties, specially its application in the general case of f_j. So, further on, we shall investigate just the class of linear functions

$$f_j(u) = \beta_0^{(j)} + \beta_1^{(j)} u, \quad j = 1, ..., q\,, \tag{1.4}$$

where $\beta_0^{(j)}, \beta_1^{(j)} \geq 0$. Therefore, our model will follow ARCH regime for the 'small' absolute values of white noise and GARCH regime for the others. The order (p, q) of this model is analogous to the standard GARCH model, but it represents the most general model of the conditional heteroscedasticity. Namely, for $c = 0$ the Split-ARCH model has the GARCH structure and, moreover, for $q = 0$ model becomes the standard ARCH model.

The constant $c > 0$ will be chosen as a proper *critical value for the reaction*, i.e. it will be the level which will determine which value of the noise will be statistically significant to let the inclusion of the previous value of the volatility in the autoregression sum of (1.3). As it is well known, according to the equation $m_c = E\left[I(\varepsilon_n^2 \geq c)\right] = P(\varepsilon_n^2 \geq c)$, it will be easily seen that the level c and the significant level m_c are connected. In the following, we explore a manner of the parameters estimations of Split-ARCH model which includes the estimation of critical value c, as well as the corresponding value m_c. Before that, we specify some stationarity conditions of our model, which are very important in the estimation procedure.

2 Strong stationarity

In order to prove the strong stationarity of the model, we shall follow the methodology used for standard GARCH model. We proved the set of conditions for the wide sense stationarity of the Split-ARCH model in [12]. In the following we shall show that the similar set of conditions is necessary and sufficient for the strong stationarity of our model. According to the standard GARCH procedure (see, for example, Mikosch [9]) or the Markov representation of conditional heteroscedasticity models described in Francq et al. [8], we can represent the Split–ARCH model by the stochastic difference equation of order one

$$\mathbf{Y_n} = \mathbf{W_n} + \mathbf{A_n}\mathbf{Y_{n-1}} \tag{2.1}$$

where

$$\mathbf{W_n} = \begin{pmatrix} \alpha_0 + \sum_{j=1}^{q} \omega_{n-j} & 0 & \cdots & 0 \end{pmatrix}', \quad \mathbf{Y_n} = \begin{pmatrix} \sigma_n^2 & \sigma_{n-1}^2 & \cdots & \sigma_{n-r+1}^2 \end{pmatrix}',$$

$$\mathbf{A_n} = \begin{pmatrix} \psi_{n-1} & \psi_{n-2} & \cdots & \psi_{n-r+1} & \psi_{n-r} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

and

$$\omega_{n-j} = \beta_0^{(j)} I(\varepsilon_{n-j}^2 \geq c), \quad \psi_{n-j} = \alpha_j\, \varepsilon_{n-j}^2 + \beta_1^{(j)} I(\varepsilon_{n-j}^2 \geq c), \quad j = 1, \ldots, r$$

$$\alpha_{p+i} = \beta_0^{(q+j)} = \beta_1^{(q+j)} = 0, \qquad i = 1, \ldots, r-p, \quad j = 1, \ldots, r-q$$

where we denoted $r = \max\{p, q\}$.

Theorem 2.1 *Let the model, Split–ARCH, be defined by the equations* (1.2), (1.3) *and* (1.4). *Then, the following conditions are equivalent:*
(i) *The polynomial* $P(\lambda) = \lambda^r - \sum_{j=1}^{r} \gamma_j\, \lambda^{r-j}$, *where*

$$r = \max\{p, q\}, \quad \gamma_j = \begin{cases} \alpha_j + m_c \beta_1^{(j)}, & 1 \leq j \leq \min\{p, q\} \\ \alpha_j, & q < j \leq p \\ m_c \beta_1^{(j)}, & p < j \leq q \end{cases},$$

has the roots $\lambda_1, ..., \lambda_r$ *which satisfy the condition*

$$|\lambda_j| < 1, \quad \forall j = 1, ..., r. \tag{2.2}$$

(ii) *The equation* (2.1) *has the unique, strong stationary and ergodic solution in the form*

$$\mathbf{Y_n} = \mathbf{W_n} + \sum_{k=1}^{\infty} \mathbf{A_n A_{n-1}} \cdots \mathbf{A_{n-k+1} W_{n-k}}. \tag{2.3}$$

(iii) $$\sum_{j=1}^{r} \gamma_j = \sum_{i=1}^{p} \alpha_i + m_c \sum_{j=1}^{q} \beta_1^{(j)} < 1.$$

(iv) *The top Lyapunov exponent*

$$\gamma = \inf_{n \in \mathbb{N}} E\left(\frac{1}{n} \ln \|\mathbf{A_n A_{n-1}} \cdots \mathbf{A_1}\|\right)$$

is strictly negative.

Proof. (i) $\Rightarrow$ (ii): Following Francq et al. [8] we introduce, for $n, k \in D$, the sequence of random vectors $\mathbf{H_k(n)} \in \mathbb{R}^r$ by the equality

$$\mathbf{H_k(n)} = \begin{cases} \mathbf{0}, & k < 0 \\ \mathbf{W_n} + \mathbf{A_n H_{k-1}(n-1)}, & k \geq 0. \end{cases}$$

Then, we have

$$\mathbf{H_k(n)} - \mathbf{H_{k-1}(n)} = \begin{cases} \mathbf{0}, & k<0 \\ \mathbf{W_n}, & k=0 \\ \mathbf{A_n}\big[\mathbf{H_{k-1}(n-1)} - \mathbf{H_{k-2}(n-1)}\big], & k>0 \end{cases}$$

and, for all $k>0$,

$$E\left\|\mathbf{H_k(n)} - \mathbf{H_{k-1}(n)}\right\| = E\left\|\mathbf{A_n}\cdots\mathbf{A_{n-k+1}}\mathbf{W_{n-k}}\right\| = \mathbf{1}_{1\times r}\cdot\mathbf{A^k}\cdot\mathbf{W} \quad (2.4)$$

where

$$\mathbf{A} = E(\mathbf{A_n}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{r-1} & \gamma_r \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$\mathbf{W} = E(\mathbf{W_n}) = \Big(\ \alpha_0 + \ m_c \sum_{j=1}^{q} \beta_0^{(j)} \quad 0 \quad \cdots \quad 0 \ \Big)'.$$

After some computation it will be seen that

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (-1)^p P(\lambda),$$

it means that the eigenvalues of the matrix $\mathbf{A}$ are the roots of the characteristics polynomial $P(\lambda)$. According to the assumption (2.2) and the equality (2.4) we have

$$E\left\|\mathbf{H_k(n)} - \mathbf{H_{k-1}(n)}\right\| \longrightarrow 0, \quad k \longrightarrow \infty$$

i.e. the sequence $\mathbf{H_k(n)}$ converges almost sure, as $k \longrightarrow \infty$, to the limit $\mathbf{Y_n}$, defined by (2.3). Since, for all fixed $k \in D$, the $\mathbf{H_k(n)}$ is strong stationary sequence, the limit $\mathbf{Y_n}$ is also strong stationary, for all $n \in D$.

(ii) $\Rightarrow$ *(iii)*: According to the equality (2.3) we have

$$E(\mathbf{Y_n}) = (\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{W} = \left(\alpha_0 + m_c \sum_{j=1}^{q} \beta_0^{(j)}\right)\left(1 - \sum_{i=1}^{p} \alpha_i - m_c \sum_{j=1}^{q} \beta_1^{(j)}\right)^{-1} \mathbf{1}_{r\times 1}$$

and, therefore, the time series (h_n^2) has the mean

$$E\left(h_n^2\right) = E\left(\sigma_n^2\right) = \left(\alpha_0 + m_c \sum_{j=1}^{q} \beta_0^{(j)}\right)\left(1 - \sum_{i=1}^{p} \alpha_i - m_c \sum_{j=1}^{q} \beta_1^{(j)}\right)^{-1}. \tag{2.5}$$

From that, as $\alpha_0 > 0$ and $\beta_0^{(j)} \geq 0,\ j = 1, \dots, q$, it will be

$$1 - \sum_{i=1}^{p} \alpha_i - m_c \sum_{j=1}^{q} \beta_1^{(j)} > 0$$

and that is obviously (iii).

$(iii) \Rightarrow (i)$: Let $\mathcal{S}_r(\mathbf{A}) = \max_j \{\lambda_j\}$ the spectral radius of the matrix $\mathbf{A}$, defined in $(i) \Rightarrow (ii)$. Then

$$\mathcal{S}_r(\mathbf{A}) \leq ||\mathbf{A}||$$

where we may set

$$||\mathbf{A}|| = \max\left\{\sum_{j=1}^{r} \gamma_j,\, 1\right\} = 1.$$

If we suppose that $\mathcal{S}r(\mathbf{A}) = 1$, then for some $\varphi \in [0, 2\pi)$ there exists an eigenvalue $\lambda' = e^{i\varphi}$ which satisfies

$$P(\lambda') = e^{ir\varphi} - \sum_{j=1}^{r} \gamma_j\, e^{i(r-j)\varphi} = 0\,.$$

After that, according to

$$\left|\, e^{ir\varphi}\right| \leq \sum_{j=1}^{r} \gamma_j \,\left| e^{i(r-j)\varphi}\right| \,,$$

it will be $\sum\limits_{j=1}^{r} \gamma_j \geq 1$, which contradicts (iii). So, $\mathcal{S}_r(\mathbf{A}) < 1$ and according to the above, it is equivalent to (i).

$(ii) \Leftrightarrow (iv)$: As $\{(A_n, W_n), n \in D\}$ is strong stationary and ergodic sequence and

$$\begin{aligned} E\ln^+\|W_0\| &= E\ln^+\left\|\alpha_0 + \sum_{j=1}^{q}\omega_{-j}\right\| \\ &\le E\ln^+\left(\|\alpha_0\| + \sum_{j=1}^{q}\left\|\beta_0^{(j)} I(\varepsilon_{-j}^2 \ge c)\right\|\right) < \infty \end{aligned}$$

and

$$\begin{aligned} E\ln^+\|A_0\| &= E\ln^+\left(\max\{1, \sum_{j=1}^{r}\psi_{-j}\}\right) \\ &\le E\ln^+\left(1 + \sum_{j=1}^{r}\alpha_j\varepsilon_{-j}^2 + \beta_1^{(j)} I(\varepsilon_{-j}^2 \ge c)\right) \\ &\le E\left(\sum_{j=1}^{r}\alpha_j\varepsilon_{-j}^2 + \beta_1^{(j)} I(\varepsilon_{-j}^2 \ge c)\right) < \infty, \end{aligned}$$

based on the well known facts from Brandt [4] or Bougerol and Picard [3], the condition $\gamma < 0$ is equivalent to (ii). □

Remark. The correlation function $\rho(k) = Corr\left(h_n^2, h_{n+k}^2\right)$ of the wide sense stationary sequence (h_n^2) will be calculated from the relation

$$\rho(k) = \frac{R(k) - E(h_n^2)^2}{R(0) - E(h_n^2)^2}, \quad k \ge 0,$$

where

$$R(k) = E(h_n^2 h_{n+k}^2) = E(h_n^2) + \sum_{j=1}^{M}\gamma_j\, R(k-j),\ R(-k) = R(k),\ R(0) = E(h_n^4).$$

Therefore, the correlation function $\rho(k)$ satisfies the difference equation

$$\rho(k) = \sum_{j=1}^{r}\gamma_j\,\rho(k-j), \quad k \ge r \tag{2.6}$$

with the initial conditions

$$\rho(0) = 1, \quad \rho(k) - \sum_{j=1}^{r}\gamma_j\,\rho(k-j) = 0, \quad 0 < k < r.$$

The relation (2.6) may be very useful in the parameters estimation procedure, like in Baillie and Chung [1], where the estimation procedure of the models of standard GARCH type is described. In the next part of this paper we shall analyze that. We shall solve the problem of estimation parameters of Split-ARCH.

3 Estimation

We shall suppose that the unknown parameters of Split-ARCH model can be defined so that they belong to the two vector sets. Let $\theta_A = (\alpha_0, \alpha_1, \ldots, \alpha_r)' \in \mathbb{R}^{r+1}$ belongs to the set

$$\Theta_A = \Big\{ \theta_A \,\Big|\, \alpha_0 > 0,\ \alpha_j \geq 0,\ j = 1, \ldots, r \ \wedge \ \sum_{j=1}^{r} \alpha_j < 1 \Big\}$$

which is the available set of parameters subject to the stationarity condition of standard ARCH models. Similarly, if we use the notation $\theta_B = (b_0, b_1, ..., b_r)' \in \mathbb{R}^{r+1}$, where $b_0 = \alpha_0 + \sum_{j=1}^{r} \beta_0^{(j)}, \quad b_j = \alpha_j + \beta_1^{(j)},\ j = 1, \ldots, r$ we shall define the open set

$$\Theta_B = \Big\{ \theta_B \,\Big|\, b_0 > 0,\ b_j \geq 0,\ j = 1, \ldots, r \ \wedge \ \sum_{j=1}^{r} b_j < 1 \Big\}$$

and remark that Θ_B is subset of the parameters set on which the strong stationarity condition of Split-ARCH model is fulfilled (Theorem 2.1).

Further on, we shall be able to estimate θ_A and θ_B as follows. As the beginning of the estimation procedure, we shall stratify the sample $S_N = \Big\{ h_t \,\Big|\, t = 1, \ldots, N \Big\}$ according to the sets

$$\begin{aligned} A_N(c) &= \Big\{ h_t \,|\, \varepsilon_{t-j}^2 < c, \quad t = r+1, \ldots, N, \quad j = 1, \ldots, r \Big\} \\ B_N(c) &= \Big\{ h_t \,|\, \varepsilon_{t-j}^2 \geq c, \quad t = r+1, \ldots, N, \quad j = 1, \ldots, r \Big\} \\ C_N(c) &= S_N \setminus \big(A_N(c) \cup B_N(c) \big). \end{aligned}$$

We can remark that Split–ARCH model will obey the ARCH structure on the data set $A_N(c)$, i.e.

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{r} \alpha_j h_{t-j}^{\,2}.$$

Now, we use the conditional least squares (CLS) method and minimize the sum

$$Q_N(\theta_A) = \sum_{h_t \in A_N(c)} \left[h_t^2 - E\left(h_t^2 \,|\mathcal{F}_{t-1} \right) \right]^2 . \tag{3.1}$$

Because of $\sigma_t^2 = E\left(h_t^2 \mid \mathcal{F}_{t-1}\right)$, the estimator $\hat{\theta}_A$ can be interpreted as a value which will minimize the objective function

$$Q_N(\theta_A) = \sum_{h_t \in A_N(c)} \Big(h_t^2 - \alpha_0 - \sum_{j=1}^{r} \alpha_j h_{t-j}^2 \Big)^2$$

and, in that way, the estimator will be any solution of the system of equations

$$\frac{\partial Q_N(\theta_A)}{\partial \alpha_j} = 0, \qquad j = 0, \ldots, r.$$

Then, the estimator $\hat{\theta}_A$ can be written in the form

$$\hat{\theta}_A = \begin{pmatrix} N_1 - r & \sum h_{t-1}^2 & \cdots & \sum h_{t-r}^2 \\ \sum h_{t-1}^2 & \sum h_{t-1}^4 & \cdots & \sum h_{t-1}^2 h_{t-r}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \sum h_{t-r}^2 & \sum h_{t-1}^2 h_{t-r}^2 & \cdots & \sum h_{t-r}^4 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \sum h_t^2 \\ \sum h_t^2 h_{t-1}^2 \\ \vdots \\ \sum h_t^2 h_{t-r}^2 \end{pmatrix}$$

where $N_1 = \operatorname{card} A_N(c)$, and all the summations are subject to t such that $h_t \in A_N(c)$. It is easy to show that in the case of Gaussian noise $\varepsilon_n : \mathcal{N}(0,1)$ the estimator $\hat{\theta}_A$ is, also, the quasi maximum likelihood estimator, described in Engle [6] or Mikosch [9]. Therefore,

$$\hat{\theta}_A = \min_{\theta_A \in \Theta_A} Q_N(\theta_A) = \max_{\theta_A \in \Theta_A} L_N(\theta_A)$$

where $L_N(\theta_A)$ is the log-likelihood function of stratum variables $h_t \in A_N(c)$.

On the other hand, the elements of the set $B_N(c)$ satisfy the relation

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{r} \left[\beta_0^{(j)} + \left(\alpha_j \; \varepsilon_{t-j}^2 + \beta_1^{(j)} \right) \sigma_{t-j}^2 \right]$$

meaning that the model is of GARCH type. So, a common way of estimating parameters is some iterative method, like Newton–Raphson's procedure (see, for instance [13]) or methods based on the minimum distance estimators of the model's autocorrelation as in Baillie and Chung [1]. Meanwhile, instead of that, we can use the maximum likelihood estimator for the elements of the volatility sequence $\hat{\sigma}_t^2 = h_t^2, \quad 1 \leq t \leq N$. After that, determine the regression coefficients θ_B applying the least squares optimization procedure on the specified sum

$$Q'_N(\gamma_1, \ldots, \gamma_r) = \sum_{h_t \in B_N(c)} \Big(h_t^2 - b_0 - \sum_{j=1}^{r} b_j h_{t-j}^2 \Big)^2. \tag{3.2}$$

This implies

$$\hat{\theta}_B = \begin{pmatrix} N_2 - r & \sum h_{t-1}^2 & \cdots & \sum h_{t-p}^2 \\ \sum h_{t-1}^2 & \sum h_{t-1}^4 & \cdots & \sum h_{t-1}^2 h_{t-p}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \sum h_{t-p}^2 & \sum h_{t-1}^2 h_{t-p}^2 & \cdots & \sum h_{t-p}^4 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \sum h_t^2 \\ \sum h_t^2 h_{t-1}^2 \\ \vdots \\ \sum h_t^2 h_{t-p}^2 \end{pmatrix}$$

where $N_2 = \operatorname{card} B_N(c)$ and the summations are subject to t such that $h_t \in B_N(c)$. Finally, the estimates $\hat{\theta}_A$ and $\hat{\theta}_B$ imply

$$\Big(\ \textstyle\sum_{j=1}^{r} \hat{\beta}_0^{(j)}, \ \hat{\beta}_1^{(1)}, \ \ldots, \ \hat{\beta}_1^{(r)} \ \Big)' = \hat{\theta}_A - \hat{\theta}_B$$

where the estimates $\hat{\beta}_0^{(1)}, \ldots, \hat{\beta}_0^{(r)}$ can be computed by the same CLS methods, using the previously obtain estimates and the following stratification of the stratum $C_N(c)$:

$$\begin{aligned} C_N^{(j)}(c) &= \Big\{ h_t \mid \varepsilon_{t-j}^2 \geq c, \ \ \varepsilon_{t-k}^2 < c, \ \ k = 1, \ldots, r, \ \ k \neq j \Big\}, \quad 1 \leq j \leq r \\ C_N^{(r+1)}(c) &= C_N(c) \setminus \bigcup_{j=1}^{r} C_N^{(j)}. \end{aligned}$$

In the next proposition we inquire the asymptotic properties of this "two–step" procedure.

Theorem 3.1 *Let, for some $N_0 > 0$ and all $N \geq N_0$, is $\hat{\theta}_A \in \Theta_A$ and $\hat{\theta}_B \in \Theta_B$. Then, the estimators $\hat{\theta}_A$ and $\hat{\theta}_B$ are strictly consistent and asymptotically normally distributed estimators for θ_A and θ_B, respectively.*

Proof. Let $\theta_A^{(0)}$ be the true value of the unknown parameter θ_A and let us set the sequence $\upsilon_t = h_t^2 - \sigma_t^2, \quad t = 1, \ldots, N$. Then, we have

$$E\left(\upsilon_t \mid \mathcal{F}_{t-1}\right) = E\left(h_t^2 \mid \mathcal{F}_{t-1}\right) - \sigma_t^2 = 0$$

i.e. (υ_t) is a martingale difference and so it is the sequence of the uncorrelated random variables. Furthermore, on the set $A_N(c)$ is

$$h_t^2 = \sigma_t^2 + \upsilon_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j h_{t-j}^2 + \upsilon_t$$

and (h_t^2) is the autoregressive time series with a white noise (υ_t). Therefore, we can use this representation to compute the spectral density of h_t^2:

$$f(\omega) = \frac{Var(\upsilon_t)}{2\pi} \prod_{j=1}^{p} \frac{1}{1 - 2\lambda_j cos\omega + \lambda_j^2}$$

and it follows that

$$f(0) = \frac{Var(\upsilon_t)}{2\pi} \prod_{j=1}^{p} \frac{1}{(1-\lambda_j)^2}\,.$$

Because of $|\lambda_j| < 1$ for all $\theta_A \in \Theta_A$, the function $f(\omega)$ is continuous for $\omega = 0$. Then, (h_t^2), as well as (υ_t), is ergodic and stationary sequence of random variables. Now, according to the Taylor expansion of $\partial Q_N / \partial \theta_A$ around $\theta_A = \theta_A^{(0)}$, we have

$$\frac{\partial Q_N(\theta_A)}{\partial \theta_A} = \frac{\partial Q_N(\theta_A^{(0)})}{\partial \theta_A} + \frac{\partial^2 Q_N(\theta_A^{(0)})}{\partial \theta_A \partial \theta_A'} \cdot (\theta_A - \theta_A^{(0)}).$$

and, substituting θ_A with $\hat{\theta}_A$ and setting $\partial Q_N(\hat{\theta}_A)/\partial \theta_A = 0$, we have

$$\hat{\theta}_A - \theta_A^{(0)} = -\left[\frac{\partial^2 Q_N(\theta_A^{(0)})}{\partial \theta_A \partial \theta_A'}\right]^{-1} \cdot \frac{\partial Q_N(\theta_A^{(0)})}{\partial \theta_A}. \tag{3.3}$$

On the other hand,

$$\frac{N_1}{N} = \frac{1}{N} \sum_{h_t \in A_N(c)} \prod_{j=1}^{r} I\left(\varepsilon_{t-j}^2 < c\right) \xrightarrow{a.s.} [F(c)]^r\,, \quad N \to \infty$$

where $F(c) = P\left(\varepsilon_t^2 < c\right) < \infty$. Then, $N_1 \xrightarrow{a.s.} \infty$, when $N \to \infty$, and we may apply the ergodic theorem on the random sums of sequences (υ_t) and

(h_t^2) (see, for instance Embrechts et al. [5]). So, we shall have the following almost sure convergence

$$\frac{1}{-2(N_1-r)} \cdot \frac{\partial Q_N(\theta_A^{(0)})}{\partial \theta_A} = \begin{pmatrix} \frac{1}{N_1-r}\sum \upsilon_t \\ \frac{1}{N_1-r}\sum \upsilon_t h_{t-1}^2 \\ \vdots \\ \frac{1}{N_1-r}\sum \upsilon_t h_{t-r}^2 \end{pmatrix} \xrightarrow{a.s.} 0, \quad \text{when} \quad N \to \infty$$

and also

$$2(N_1-r)\left[\frac{\partial^2 Q_N(\theta_A^{(0)})}{\partial\theta_A \partial\theta_A'}\right]^{-1} =$$

$$= \begin{pmatrix} 1 & \frac{1}{N_1-r}\sum h_{t-1}^2 \cdots \frac{1}{N_1-r}\sum h_{t-r}^2 \\ \frac{1}{N_1-r}\sum h_{t-1}^2 & \frac{1}{N_1-r}\sum h_{t-1}^4 \cdots \frac{1}{N_1-r}\sum h_{t-1}^2 h_{t-r}^2 \\ \vdots & \vdots \qquad\qquad \vdots \\ \frac{1}{N_1-r}\sum h_{t-r}^2 & \frac{1}{N_1-r}\sum h_{t-1}^2 h_{t-r}^2 \cdots \frac{1}{N_1-r}\sum h_{t-r}^4 \end{pmatrix}^{-1} \xrightarrow{a.s.} \Gamma^{-1}$$

where $h_t \in A_N(c)$ and the second-moment matrix $\Gamma = E\left(\mathbf{h_t h_t'}\right)$, where $\mathbf{h_t} = \left(1, h_{t-1}^2, \ldots, h_{t-r}^2\right)'$, does not depend on t for all θ_A that satisfy the stationarity condition. That means for all $\theta_A \in \Theta_A$. These two convergences yield, with probability one,

$$\hat{\theta}_A - \theta_A^{(0)} \to 0, \quad N \to \infty$$

i.e. the estimator $\hat{\theta}_A$ is strictly consistent.

Now, we shall show the asymptotic normality of $\hat{\theta}_A$. Using the representation (3.3) we can write

$$\sqrt{N_1 - r}\,\left(\hat{\theta}_A - \theta_A^{(0)}\right) = \mathbf{U}_N^{-1} \cdot \mathbf{V}_N$$

where

$$\mathbf{U}_N = \frac{1}{2(N_1-r)} \cdot \frac{\partial^2 Q_n(\theta_A^{(0)})}{\partial\theta_A \partial\theta_A'}, \qquad \mathbf{V}_N = \frac{-1}{2\sqrt{N_1-r}} \cdot \frac{\partial Q_N(\theta_A^{(0)})}{\partial\theta_A}.$$

For any nonzero constant vector $\mathbf{c} = (c_0, \ldots, c_r)' \in \mathbb{R}^{r+1}$ the random sequence

$$\sqrt{N_1 - r}\,\mathbf{c}'\mathbf{V}_N = \sum_{h_t \in A_N(c)} \upsilon_t\Big(c_0 + \sum_{j=1}^{r} c_j h_{t-j}^2\Big)$$

is a martingale and, according to the Billingsley's central limit theorem for martingales (see for instance [10]), we have:

$$\mathbf{c}'\mathbf{V}_N \xrightarrow{d} \mathcal{N}(0, \mathbf{c}'\Lambda\mathbf{c})$$

where $\Lambda = E(\mathbf{g_t}\mathbf{g_t'})$, $\quad \mathbf{g_t} = \upsilon_t \left(1, h_{t-1}^2, \ldots, h_{t-r}^2\right)'$ and Λ does not depend on t. Using this convergence and Cramer-Wald's decomposition, we have

$$\mathbf{V}_N \xrightarrow{d} \mathcal{N}(0, \Lambda)$$

and, because of $\mathbf{U}_N^{-1} \xrightarrow{a.s.} \Gamma^{-1}$ when $N \to \infty$, we got finally

$$\sqrt{N_1 - r}\,\left(\hat{\theta}_A - \theta_A^{(0)}\right) \xrightarrow{d} \mathcal{N}(0, \Gamma^{-1}\Lambda\ \Gamma^{-1}).$$

In the analogue way, it can be easily proved the strong consistency and the asymptotic normality of the sequence $\hat{\theta}_B$. $\square$

4 Application

Aforesaid method of the CLS-estimating of the Split-ARCH parameters can be easily applied in the empirical analysis of the most financial sequences. In following, we present the results of estimation of log-returns of the price of precious metals, upon the Kitco Inc. data and the database London Fix in the period 1998-2005. Like a comparatione, we estimated both the coefficients of standard ARCH and Split-ARCH model, taking the most simple case of those models. The results of the estimation of the parameters of ARCH(1) model

$$h_n = \sigma_n \varepsilon_n, \qquad \sigma_n^2 = \alpha_0 + \alpha_1 h_{n-1}^2, \qquad n \in D$$

are presented in Table 1.

We presented the size of the observed time series in the first row of the table and then we showed the realized values based on the Lagrange multiplier (LM) test, introduced by Engle [6]. The most precisely, these values show the asymptotic values of LM statistics, which can be calculated as NR^2, where N is the size of the observed series and R^2 is the coefficient of determination (square of the multicorrelation coefficient), calculated by the regression procedure of the parameters' estimation. As Engle [6] showed, LM statistic is asymptotically χ_p^2 distributed (p is the order of ARCH model), under the null hypotheses that there is no ARCH effect in the observed data, i.e.

$$H_0 : \alpha_1 = \cdots = \alpha_p = 0.$$

Table 1: Estimated values of ARCH(1) model

Parameters	Precious metals			
	Gold	**Silver**	**Palladium**	**Platinum**
N	1 841	1 803	1 668	1 705
LM	64.745	97.645	51.872	52.392
$\hat{\alpha}_0$	$7.57 \cdot 10^{-5}$	$2.03 \cdot 10^{-4}$	$5.32 \cdot 10^{-4}$	$1.65 \cdot 10^{-4}$
$\hat{\alpha}_1$	0.1875	0.2327	0.1762	0.1752
SEE	$3.55 \cdot 10^{-4}$	$6.82 \cdot 10^{-4}$	$1.68 \cdot 10^{-4}$	$4.70 \cdot 10^{-4}$
$\hat{\rho}$	75.79%	75.68%	92.02%	85.98%

It is easily seen that, in our case, the obtained values of LM statistics indicate that all the observed time series have the emphatic ARCH effect and the given results in parameters' estimation are adequate. In fact, the estimated values of the parameters α_1 satisfied the stationarity condition $|\alpha_1| < 1$ and we may use the standard procedure in ARCH modeling of those empirical data sets. We can see the quality of that modeling in the last two rows of the table above. We showed the standard errors of the estimation (SEE) there, as the estimated values of the correlation coefficients ($\hat{\rho}$) between the empirical and the ARCH-modeled data, respectively. In Figure 2, the degree of the correlation of the ARCH(1)-modeled values with the log-returns of the gold's prices can be seen.

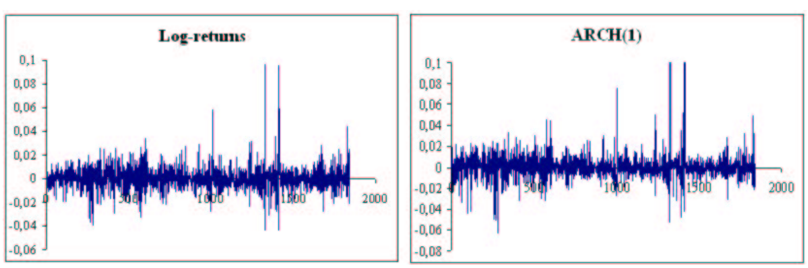

Figure 2: Comparative illustrations for the original data (left) and ARCH(1) modeled gold data (right).

After that, the same data set was processed in two–step Split–ARCH modeling scheme. The stratification was done using the estimates

$$\hat{\varepsilon}_t = h_t / \hat{\sigma}_t, \quad t = 1, \ldots, N,$$

where $\hat{\sigma}_t$ was the empirical standard deviation of the sample (see Figure 3). As a critical value for the reaction, we used the mean value of the χ_1^2 distribution, i.e. $c = E(\varepsilon_t^2) = 1$ and that was the starting point for the following Split–ARCH estimation of the real data set.

Table 2: Estimated values of Split-ARCH(1,1) model

Parameters	**Gold**	**Silver**	**Palladium**	**Platinum**
		I stratum		
N_1	1 342	1 413	1 248	1 188
LM	5.138	5.453	12.536	17.209
$\hat{\alpha}_0$	$5.90 \cdot 10^{-5}$	$6.57 \cdot 10^{-5}$	$1.28 \cdot 10^{-4}$	$9.51 \cdot 10^{-5}$
$\hat{\alpha}_1$	0.0619	0.0621	0.0040	0.1210
SEE	$2.21 \cdot 10^{-5}$	$8.10 \cdot 10^{-5}$	$1.42 \cdot 10^{-5}$	$4.47 \cdot 10^{-5}$
		II stratum		
N_2	499	390	420	517
LM	16.264	19.268	5.249	20.129
$\hat{\gamma}_0$	$9.14 \cdot 10^{-5}$	$7.59 \cdot 10^{-4}$	$1.94 \cdot 10^{-4}$	$4.47 \cdot 10^{-4}$
$\hat{\gamma}_1$	0.1805	0.2222	0.1118	0.1975
SEE	$6.44 \cdot 10^{-4}$	$1.25 \cdot 10^{-3}$	$2.87 \cdot 10^{-4}$	$7.38 \cdot 10^{-4}$
$\hat{\rho}$	97.85%	85.07%	97.97%	91.94%

We can see the estimated values of the Split–ARCH parameters in Table 2. The simple comparatione of the displayed values can give the explanation why to proceed the dynamic of the prices of precious metals by Split–ARCH. First of all, the considered real data have greater the correlation coefficient to the values modelled by Split–ARCH than to the ARCH(1). Also, it can be seen that the fluctuation of Split–ARCH values is more likely the real data values than when comparing ARCH approximation (see Figure 3).

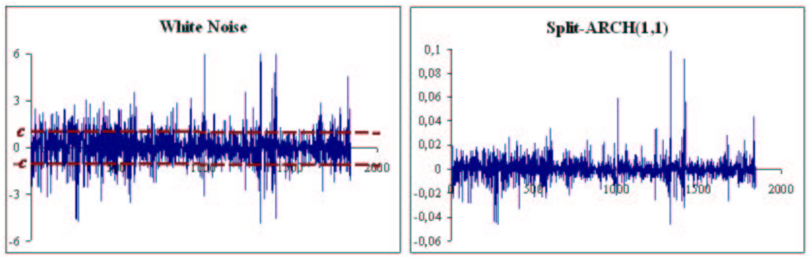

Figure 3: Stratification of the residuals of log-returns (left) and Split-ARCH modelled gold data set (right).

Finally, we shall investigate the prediction of the future values of log–returns (h_n) in two cases: when follow the time series model of ARCH and Split-ARCH type. Let us predict h_{n+m} according to the finite part of the one realization of log-returns: $h_1, \ldots, h_n$. The optimal prediction according to the minimum mean square error procedure will be trivial as (h_n) is the sequence of martingale differences,

$$\hat{h}_{n+m} = E\left(h_{n+m} \middle| \mathcal{F}_n\right) = 0\,.$$

But, if we predict a nonlinear function of h_{n+m} instead, as for instance is h^2_{n+m}, we shall have

$$\hat{h}^2_{n+m} = E\left(h^2_{n+m} \middle| \mathcal{F}_n\right) = E\left[\sigma^2_{n+m}\, E\left(\varepsilon^2_{n+m} \middle| \mathcal{F}_{n+m-1}\right) \middle| \mathcal{F}_n\right] = \hat{\sigma}^2_{n+m}$$

and our task is changed in the sense that we should predict volatility σ^2_{n+m}. Meanwhile, (2.1) implies

$$\mathbf{Y_{n+m}} = \mathbf{W_{n+m}} + \sum_{k=1}^{m-1} \mathbf{A_{n+m}} \ldots \mathbf{A_{n+m-k+1}} \mathbf{W_{n+m-k}} + \left(\prod_{k=1}^{m} \mathbf{A_{n+m-k+1}} \right) \mathbf{Y_n}$$

and the corresponding estimator for $\mathbf{Y}_{n+m}$ will be

$$\widehat{\mathbf{Y}}_{\mathbf{n+m}} = E\left(\mathbf{Y_{n+m}} \middle| \mathcal{F}_n\right) = \left(\mathbf{I} - \mathbf{A^m}\right)\left(\mathbf{I} - \mathbf{A}\right)^{-1} \mathbf{W} + \mathbf{A^m} \mathbf{Y_n}. \qquad (4.1)$$

In the simplest case of Split-ARCH time series, where $p = q = 1$, equation (4.1) expresses the prediction of volatility

$$\hat{\sigma}^2_{n+m} = \gamma_0 \frac{1 - \gamma_1^m}{1 - \gamma_1} + \gamma_1^m h_n^2$$

and this equality becomes most simple for ARCH(1) model, just needs to substitute γ_0, γ_1 with α_0, α_1, respectively. Now, those formulas can be used when solving some practical problem in the prediction of the empirical data sets. Let

$$S_{n+m} = S_n e^{h_{n+1}+\cdots+h_{n+m}}$$

according to (1.1) be a price at some future moment $n+m$. If the limit distribution of (h_n) is normal, then, because of

$$E\left[(h_{n+1}+\ldots h_{n+m})^2 \,\middle|\, \mathcal{F}_n\right] = \sum_{k=1}^{m} E\left(h_{n+k}^2 \middle| \mathcal{F}_n\right) = \sum_{k=1}^{m} \widehat{\sigma}_{n+k}^2$$

the $(1-\alpha)\%$–confidence interval for the future price S_{n+m} is

$$S_n\, e^{-z_{\alpha/2}\sqrt{\sum_{k=1}^{m}\widehat{\sigma}_{n+k}^2}} \le S_{n+m} \le S_n\, e^{z_{\alpha/2}\sqrt{\sum_{k=1}^{m}\widehat{\sigma}_{n+k}^2}}, \tag{4.2}$$

where $\displaystyle\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z_{\alpha/2}} e^{-x^2/2}dx = 1-\alpha/2.$

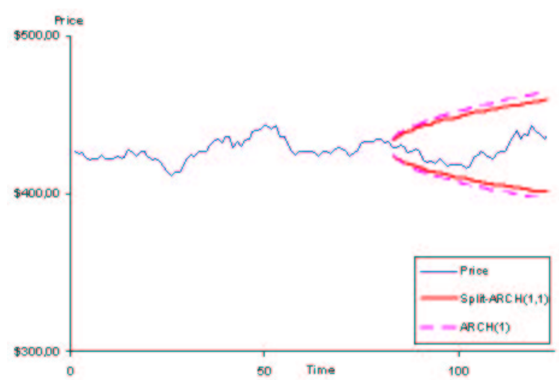

Figure 4: Extrapolation of the price of gold according to ARCH and Split-ARCH representation.

We displayed the application of the this procedure in Figure 4. The London Fix market's data between the 1st of May and the 30th of June 2005. were predicted, where we used the following procedure (4.2) and 90%-confidence interval for the price. The data were fitted by ARCH(1) and

Split-ARCH(1,1) time series model defined as above. Let us emphasize that the price value is inside the both of displayed confidence intervals, but the Split-ARCH prediction is somewhat better regarding the adequate ARCH prediction.

5 Conclusion

We applied the Split–ARCH(1,1) model which we defined for the soybean meal from the Product Exchange Novi Sad and for the oil price according to Wall Street Journal in [12]. Also, we applied it to the real data of the prices of precious metals according to the Kitco Inc. data and the database London Fix. Here we estimated parameters of Split–ARCH in the way that is common for all ARCH type models. We found out that the Split–ARCH is more convenient for this data. We compare the correlation coefficients of the models and the real data sets. It is more efficient in prediction also and we demonstrated that using the price of gold on London Fix market.

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