Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **21:2** (2007), 153–160

DIVERGENT CESÀRO MEANS OF FOURIER EXPANSIONS WITH RESPECT TO POLYNOMIALS ASSOCIATED WITH THE MEASURE $(1-x)^{\alpha}(1+x)^{\beta}+M\Delta_{-1}$

Bujar Xh. Fejzullahu

Abstract

We prove that, for certain indices of δ, there are functions whose Cesàro means of order δ of the Fourier expansion with respect to the polynomials associated with the measure $(1-x)^{\alpha}(1+x)^{\beta}+M\Delta_{-1}$, where Δ_t is the delta function at a point t, are divergent almost everywhere on $[-1,1]$. We follow Meaney's paper (2003), where divergent Cesàro and Riesz means of Jacobi expansions were proved.

1 Introduction

Let $d\mu$ be a finite positive Borel measure on the interval $I \subset R$ such that *supp* $(d\mu)$ is an infinite set and let $p_n(d\mu)$ denote the corresponding orthonormal polynomials. For $f \in L^1(I, d\mu)$, let $S_n f$ denote the nth partial sum of the orthonormal Fourier expansion of f in $\{p_n(d\mu)\}_{n=0}^{\infty}$:

$$S_N f(x) = \sum_{n=0}^{N} c_n(f) p_n(x), \tag{1}$$

$$c_n(f) = \int_I f p_n d\mu.$$

The Cesàro means of order δ of the expansion (1) are defined by

$$\sigma_N^{\delta} f(x) = \sum_{n=0}^{N} \frac{A_{N-n}^{\delta}}{A_N^{\delta}} c_n(f) p_n(x),$$

2000 *Mathematics Subject Classification.* 42C05, 42C10.
Key words and phrases. Koornwinder's Jacobi-type polynomials, Cesàro mean.
Received: May 22, 2007

where $A_k^\delta = \binom{k+\delta}{k}$.
The study of the convergence of Fourier series (1) with respect to polynomials associated with the measure $d\mu = (1-x)^\alpha(1+x)^\beta dx + M\Delta_{-1} + N\Delta_1$ has been discussed in [2] (see also [4]). If $\alpha, \beta \geq -1/2$, then (see [2])

$$||S_n f||_{L^p(d\mu)} \leq C||f||_{L^p(d\mu)} \qquad \forall n \geq 0,\ \forall f \in L^p(d\mu)$$

if and only if p belongs to the open interval (p_0, p_1), where

$$p_0 = \frac{4(\alpha+1)}{2\alpha+3}, \qquad p_1 = \frac{4(\alpha+1)}{2\alpha+1}$$

when $\alpha \geq \beta$ and $\alpha > -1/2$ (and the analogous formulas with α replaced by β if $\beta \geq \alpha$).
We show that, for $1 \leq p < p_0$ and $0 \leq \delta < \frac{2\alpha+2}{p} - \frac{2\alpha+3}{2}$, there are functions whose Fourier expansions associated to a measure $d\mu = (1-x)^\alpha(1+x)^\beta dx + M\Delta_{-1}$ have almost everywhere divergent Cesàro means of order δ.

2 Koornwinder's Jacobi-type polynomials

Let $\omega_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, $(\alpha, \beta > -1)$, be the Jacobi measure on the interval $[-1,1]$. In [5] T. H. Koornwinder introduced the polynomials $\{P_n^{\alpha,\beta,M,N}(x)\}_{n=0}^\infty$ which are orthogonal on the interval $[-1,1]$ with respect to the weight function

$$d\mu(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\omega_{\alpha,\beta}(x)dx + M\Delta_{-1}(x) + N\Delta_1(x), \quad (2)$$

where $\alpha > -1$, $\beta > -1$, $M \geq 0$, and $N \geq 0$. We call these polynomials the Koornwinder's Jacobi-type polynomials.
We denote the orthonormal Koornwinder's Jacobi-type polynomial by $p_n^{(\alpha,\beta,M,N)}$, which differs from $P_n^{(\alpha,\beta,M,N)}$ by normalization constant ([9, p. 81]).
Some basic properties of $p_n^{(\alpha,\beta,M,N)}$ (see [9, Chapter IV]), we will need in the following, are given below:

$$p_n^{(\alpha,\beta,M,N)}(1) \sim \begin{cases} n^{-\alpha-3/2} & \text{if } N > 0 \\ n^{\alpha+1/2} & \text{if } N = 0; \end{cases} \quad (3)$$

$$|p_n^{(\alpha,\beta,M,N)}(-1)| \sim \begin{cases} n^{-\beta-3/2} & \text{if } M > 0 \\ n^{\beta+1/2} & \text{if } M = 0; \end{cases} \quad (4)$$

$$|p_n^{(\alpha,\beta,M,N)}(cos\theta)| = \begin{cases} O(\theta^{-\alpha-1/2}) & \text{if } c/n \le \theta \le \pi/2, \\ O(n^{\alpha+1/2}) & \text{if } 0 \le \theta \le c/n \end{cases} \quad (5)$$

for $\alpha \ge -1/2,\ \beta \ge -1/2$, and $n \ge 1$.
Asymptotic behaviour of the Jacobi orthonormal polynomials $p_n^{(\alpha,\beta)}$, for $x \in [-1+\epsilon, 1-\epsilon]$ and $\epsilon > 0$, it is given by (see [8, Theorem 8.21.8])

$$p_n^{(\alpha,\beta)}(x) = r_n^{\alpha,\beta}(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}cos(k\theta+\gamma) + O(n^{-1}), \quad (6)$$

where $x = cos\theta$, $k = n + (\alpha+\beta+1)/2$, $\gamma = -(\alpha+1/2)\pi/2$ and $r_n^{\alpha,\beta} = \frac{2^{(\alpha+\beta+1)/2}(\pi n)^{-1/2}}{||P_n^{(\alpha,\beta)}||_2} \to (2/\pi)^{1/2}$.
Now we will show that the polynomials $p_n^{(\alpha,\beta,M,0)}$ have a similar asymptotic behaviour to the one of $p_n^{(\alpha,\beta)}(x)$.

Lemma 1. *Let $p_n^{(\alpha,\beta,M,0)}$ be the polynomials orthonormal with respect to a measure $d\mu = \omega_{\alpha,\beta}dx + M\Delta_{-1}$ and A_n, B_n the corresponding coefficients which appear in [3, Proposition 4]. Then, for $x \in [-1+\epsilon, 1-\epsilon]$ and $\epsilon > 0$, we have*

$$p_n^{(\alpha,\beta,M,0)}(x) = s_n^{\alpha,\beta}(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}cos(k\theta+\gamma) + O(n^{-1}),$$

where $s_n^{\alpha,\beta} = A_n r_n^{\alpha,\beta} + B_n r_{n-1}^{\alpha,\beta+2}$.

Proof. By [3, Proposition 4]

$$p_n^{(\alpha,\beta,M,0)}(x) = A_n p_n^{(\alpha,\beta)}(x) + B_n(x+1)p_{n-1}^{(\alpha,\beta+2)}(x). \quad (7)$$

¿From (6), we have

$$p_{n-1}^{(\alpha,\beta+2)}(x) = r_{n-1}^{\alpha,\beta+2}(1+x)^{-1}(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}cos(k\theta+\gamma)+O(n^{-1}).$$

Hence, from this and (6), we obtain

$$p_n^{(\alpha,\beta,M,0)}(x) = (1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}cos(k\theta+\gamma)$$
$$[A_n r_n^{\alpha,\beta} + B_n r_{n-1}^{\alpha,\beta+2}] + [A_n + B_n(x+1)]O(n^{-1})$$

Since

$$A_n \cong cn^{-2\beta-2}, \qquad B_n \cong 1, \quad (8)$$

see [8, p. 72, (4.5.8)] and [3, Proposition 4], where by $u_n \cong v_n$ we mean that the sequence u_n/v_n converges to 1, we get

$$p_n^{(\alpha,\beta,M,0)}(x) = s_n^{\alpha,\beta}(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}cos(k\theta+\gamma) + O(n^{-1}).$$

□

The formula of Mehler-Heine type for Jacobi orthonormal polynomials is (see [8, Theorem 8.1.1])

$$\lim_{n\to\infty} n^{-\alpha-1/2} p_n^{(\alpha,\beta)}(cos\frac{z}{n}) = c_{\alpha,\beta}\,(z/2)^{-\alpha}\,J_\alpha(z), \tag{9}$$

where $\alpha,\ \beta$ real numbers, $c_{\alpha,\beta}$ is positive constant independent of n and z, and $J_\alpha(z)$ is the Bessel function. This formula holds uniformly for $|z| \leq R$, R fixed.
Using (7), (8) and (9) we have:

Lemma 2. *Let $\alpha > -1,\ \beta > -1$ and $M > 0$. Then*

$$\lim_{n\to\infty} n^{-\alpha-1/2} p_n^{(\alpha,\beta,M,0)}(cos\frac{z}{n}) = 2c_{\alpha,\beta+2}\,(z/2)^{-\alpha}\,J_\alpha(z),$$

which holds uniformly for $|z| \leq R$, R fixed.

For every function $f \in L^1([-1,1], d\mu)$ the Fourier coefficients of the series (1) can be written as

$$c_n(f) = c'_n(f) + Mf(-1)p_n^{(\alpha,\beta,M,N)}(-1) + Nf(1)p_n^{(\alpha,\beta,M,N)}(1), \tag{10}$$

where

$$c'_n(f) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} f(x)p_n^{(\alpha,\beta,M,N)}(x)\omega_{\alpha,\beta}(x)dx.$$

We next need to know the bounds for the integral involving Koorwinder's Jacobi-type polynomials

$$\int_{-1}^{1} |p_n^{(\alpha,\beta,M,N)}(x)|^q \omega_{\alpha,\beta}(x)dx$$

where $1 \leq q < \infty$.
For $M = N = 0$ the calculation of this integral is in [8, p.391. Exercise 91] (see also [6]).
First we prove the upper bound for this integral:

Theorem 1. *Let $M \geq 0$ and $N \geq 0$. For $\alpha \geq -1/2$*

$$\int_0^1 (1-x)^\alpha |p_n^{(\alpha,\beta,M,N)}(x)|^q dx = \begin{cases} O(1) & \text{if } 2\alpha > q\alpha - 2 + q/2, \\ O(log\ n) & \text{if } 2\alpha = q\alpha - 2 + q/2, \\ O(n^{q\alpha+q/2-2\alpha-2}) & \text{if } 2\alpha < q\alpha - 2 + q/2. \end{cases}$$

Proof. ¿From (5), for $q\alpha + q/2 - 2\alpha - 2 \neq 0$, we have

$$\int_0^1 (1-x)^\alpha |p_n^{(\alpha,\beta,M,N)}(x)|^q dx = O(1)\int_0^{\pi/2} \theta^{2\alpha+1}|p_n^{(\alpha,\beta,M,N)}(cos\theta)|^q d\theta$$
$$= O(1)\int_0^{n^{-1}} \theta^{2\alpha+1} n^{q\alpha+q/2} d\theta + O(1)\int_{n^{-1}}^{\pi/2} \theta^{2\alpha+1}\theta^{-q\alpha-q/2} d\theta$$
$$= O(n^{q\alpha+q/2-2\alpha-2}) + O(1),$$

and for $q\alpha + q/2 - 2\alpha - 2 = 0$ we have

$$\int_0^1 (1-x)^\alpha |p_n^{(\alpha,\beta,M,N)}(x)|^q dx = O(log\ n).$$

□

Now using a technique similar to the one used in [8, Theorem 7.34] we obtain:

Theorem 2. *Let $M \geq 0$ and $N = 0$. For $\alpha \geq -1/2$ and $2\alpha < q\alpha - 2 + q/2$ we have*

$$\int_0^1 (1-x)^\alpha |p_n^{(\alpha,\beta,M,0)}(x)|^q dx \sim n^{q\alpha+q/2-2\alpha-2}$$

Proof. For the proof of Theorem 2 it is sufficient to prove just the lower bound for the integral.
Let $\alpha \geq -1/2$ and $M > 0$. According to Lemma 2, we have

$$\int_0^{\pi/2} \theta^{2\alpha+1}|p_n^{(\alpha,\beta,M,0)}(cos\theta)|^q d\theta > \int_0^{n^{-1}} \theta^{2\alpha+1}|p_n^{(\alpha,\beta,M,0)}(cos\theta)|^q d\theta$$
$$\cong c\int_0^1 (z/n)^{2\alpha+1} n^{q\alpha+q/2} |(z/2)^{-\alpha} J_\alpha(z)|^q\ n^{-1} dz \sim n^{q\alpha+q/2-2\alpha-2}.$$

□

3 Divergent Cesàro means of the Fourier expansion with respect to polynomials associated with the measure $(1-x)^\alpha(1+x)^\beta + M\Delta_{-1}$

In [10, Theorem 3.1.22] it is proved

Lemma 3. *Suppose that* $\lim_{N\to\infty} \sigma_N^\delta f(x)$ *exists for some* $x \in [-1,1]$ *and* $\delta > -1$. *Then*

$$|c_N(f)p_N(x)| \le C_\delta N^\delta \max_{0\le n\le N} |\sigma_n^\delta f(x)|, \qquad \forall N \ge 0.$$

From Egorov's theorem and Lemma 3 it follows that if the series (1) is Cesàro summable of order δ on a set of positive measure in $[-1,1]$ then there is a set of positive measure E on which

$$|n^{-\delta} c_n(f) p_n^{(\alpha,\beta,M,0)}(x)| \le A.$$

Hence, from Lemma 1, we have

$$|n^{-\delta} c_n(f) \left(cos(k\theta+\gamma) + O(n^{-1})\right)| \le A.$$

uniformly for $cos\theta \in E$. Using the argument of the subsection 1.5 in [7] we obtain

$$|\frac{c_n(f)}{n^\delta}| \le A, \qquad \forall n \ge 1. \tag{11}$$

From Theorem 2, for $\alpha > -1/2$ and $1 \le q < \infty$, we have

$$\left(\int_{-1}^1 |p_n^{(\alpha,\beta,M,0)}(x)|^q \omega_{\alpha,\beta}(x)dx\right)^{1/q} > c\left(\int_0^1 (1-x)^\alpha |p_n^{(\alpha,\beta,M,0)}(x)|^q dx\right)^{1/q} \sim n^{\alpha+1/2-2\alpha/q-2/q} \tag{12}$$

where $q > \frac{4(\alpha+1)}{2\alpha+1}$.
For $q = \infty$ and $\alpha \ge \beta \ge -1/2$ we have (see [9, (4.42), p.90])

$$\max_{-1\le x\le 1} |p_n^{(\alpha,\beta,M,0)}(x)| \sim p_n^{(\alpha,\beta,M,0)}(1) \sim n^{\alpha+1/2}. \tag{13}$$

Now we are in position to prove our main result:

Theorem 3. *Let given numbers α, β, p, and δ be such that $\alpha > -1/2$;*

$$-\frac{1}{2} \leq \beta \leq \alpha;$$

$$1 \leq p < \frac{4(\alpha+1)}{2\alpha+3};$$

$$0 \leq \delta < \frac{2\alpha+2}{p} - \frac{2\alpha+3}{2}.$$

There is an $f \in L^p([-1,1], \omega_{\alpha,\beta})$, supported in $[0,1]$, whose Cesàro means $\sigma_N^\delta f(x)$ is divergent almost everywhere on $[-1,1]$.

Proof. Suppose that

$$\delta < \frac{2\alpha+2}{p} - \frac{2\alpha+3}{2}.$$

For q conjugate to p, from last inequality, we get

$$\delta < \alpha + \frac{1}{2} - \frac{2\alpha}{q} - \frac{2}{q}.$$

From the argument given in [7, Subsection 1.4], (12) and (13), for linear functional $c'_n(f) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} f(x) p_n^{(\alpha,\beta,M,0)}(x)\omega_{\alpha,\beta}(x)dx$, it follows that there is an $f \in L^p([-1,1], \omega_{\alpha,\beta})$, supported on $[0,1]$, for which satisfy

$$\frac{c'_n(f)}{n^\delta} \to \infty, \qquad as\ n \to \infty.$$

Hence, from (10), we obtain

$$\frac{c_n(f)}{n^\delta} \to \infty, \qquad as\ n \to \infty.$$

Since this result is contrary with (11) it follows that for this f the $\sigma_N^\delta f(x)$ is divergent almost everywhere. □

Remark 1. *Using formulae in [1], which relate the Riesz and Cesàro means of order $\delta \geq 0$, we conclude that Theorem 3 holds for Riesz means.*

Remark 2. *From the simmetry $P_n^{(\alpha,\beta,M,0)}(-x) = (-1)^n P_n^{(\beta,\alpha,0,M)}(x)$ we get the same results as above for the measure $d\mu = \omega_{\alpha,\beta}dx + N\Delta_1$.*

References

[1] J. J. Gergen, *Summability of double Fourier series*, Duke Math. J., **3** (1937), 133-148.

[2] J. J. Guadalupe, M. Pérez, F. J. Ruiz and J. L. Varona, *Convergence in the mean of the Fourier series with respect to polynomials associated with the measure* $(1-x)^\alpha(1+x)^\beta+M\delta_{-1}+N\delta_1$, Orthogonal polynomials and their applications (Spanish), (1989), 91-99.

[3] J. J. Guadalupe, M. Pérez, F. J. Ruiz and J. L. Varona, *Asymptotic behaviour of orthogonal polynomials relative to measures with mass points*, Mathematika, **40** (1993), 331-344.

[4] J. J. Guadalupe, M. Pérez, F. J. Ruiz and J. L. Varona, *Weighted norm inequalities for polynomial expansions associated to some measures with mass points*, Constr. Approx., **12** (1996), 341-360.

[5] T. H. Koornwinder, *Orthogonal polynomials with weight function* $(1-x)^\alpha(1+x)^\beta+M\delta(x+1)+N\delta(x-1)$, Canad. Math. Bull., **27** (1984), 205-214.

[6] C. Markett, *Cohen type inequalities for Jacobi, Laguerre and Hermite expansions*, SIAM J. Math. Anal., **14** (1983), no. 4, 819-833.

[7] Ch. Meaney, *Divergent Cesàro and Riesz means of Jacobi and Laguerre expansions*, Proc. Amer. Math. Soc., **131** (2003), no. 10, 3123-3128.

[8] G. Szegő, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Pub. 23, Amer. Math. Soc., Providence, RI (1975).

[9] J. L. Varona, *Convergencia en* L^p *con pesos de la serie de Fourier respecto de algunos sistemas ortogonales*, Ph. D. Thesis, Sem. Mat. García de Galdeano, sec. 2, no. 22. Zaragoza, (1989); http://www.unirioja.es/cu/jvarona/papers.html.

[10] A. Zygmund, *Trigonometric series: Vols. I, II*, Cambridge University Press, London (1968).

Faculty of Mathematics and Sciences, University of Pristina
Mother Teresa 5, 10000 Pristina, Kosovo, Serbia

E-mail: bujarfej@uni-pr.edu