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ORBITS TENDING TO INFINITY UNDER SEQUENCES OF OPERATORS ON HILBERT SPACES

Sonja Mančevska and Marija Orovčanec

Abstract

In this paper are considered some sufficient conditions under which, for given sequence $(T_i)_{i\geq 1}$ of operators on an infinite-dimensional complex Hilbert space, there is a *dense* set of points whose orbits under each T_i tend strongly to infinity.

1 Introduction

Throughout this paper $\mathcal{H}$ will denote an infinite-dimensional complex Hilbert space with inner product $\langle\cdot|\cdot\rangle$ and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$, with $r(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ we will denote the spectral radius, the spectrum, the point spectrum (i.e. the set of all eigenvalues for T) and the approximate point spectrum of T, respectively. Recall that $\sigma_a(T)$ is the set of all $\lambda \in \sigma(T)$ for which there is a sequence of unit vectors $(x_n)_{n\geq 1}$ such that $\|Tx_n - \lambda x_n\| \to 0$, as $n \to \infty$; any sequence with this property is a sequence of *almost eigenvectors* for λ. Unlike the point spectrum, which can be empty, the approximate point spectrum is nonempty for every $T \in \mathcal{B}(\mathcal{H})$: it contains both $\partial\sigma(T) \neq \varnothing$ and $\sigma_p(T)$ [2, Prop.VII.6.7].

Orbit of $x \in \mathcal{H}$ under $T \in \mathcal{B}(\mathcal{H})$ is the sequence $\mathrm{Orb}(T, x) := \{T^n x : n \geq 0\}$. In our interest are operators $T \in \mathcal{B}(\mathcal{H})$ for which there is $x \in \mathcal{H}$ whose orbit under T tends strongly to infinity, i.e. $\|T^n x\| \to \infty$, as $n \to \infty$. If $\sigma_p(T)$

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contains a point λ with $|\lambda| > 1$, then for every nonzero vector x in the corresponding eigenspace $\ker(T-\lambda)$, $\|T^n x\| = |\lambda|^n \|x\| \to \infty$ as $n \to \infty$. But, in general, $\ker(T-\lambda)$ is not dense in $\mathcal{H}$. In order to produce a *dense* set of vectors whose orbits under T tend strongly to infinity, we have to look at the points in the approximate point spectrum which are not eigenvalues.

In [1, Thm. III.2.A.1] B. Beauzamy showed that, if $\{\lambda \in \mathbb{C} : |\lambda| = r(T)\}$ contains a point in $\sigma(T)$ which is not an eigenvalue for T, then for every positive sequence $(\alpha_n)_{n\geq 1}$ strictly decreasing to 0, in every open ball in $\mathcal{H}$ with radius strictly larger then α_1, there is $z \in \mathcal{H}$ with $\|T^n z\| \geq \alpha_n r(T)^n$, for all $n \geq 1$. As its proof suggests, this result will remain true if $r(T)$ is replaced with $|\lambda|$ for any $\lambda \in \sigma_a(T)\backslash\sigma_p(T)$. Note that, if $r(T) > 1$, or in the later case if there is $\lambda \in \sigma_a(T)\backslash\sigma_p(T)$ with $|\lambda| > 1$, then the space will contain a dense set of z's with $\mathrm{Orb}(T, z)$ tending strongly to infinity. (For some additional results on similar estimations of the orbits we refer the reader to [5], [6] and [7].)

Using a similar technique, in [4] the first author has shown the following results.

Theorem 1.1. [4, Theorem. 3.1] *If T_1 and T_2 are operators $\in \mathcal{B}(\mathcal{H})$, and $\lambda_1 \in \sigma_a(T_1)\backslash\sigma_p(T_1)$ and $\lambda_2 \in \sigma_a(T_2)\backslash\sigma_p(T_2)$, then for any two sequences $(\alpha_{1,n})_{n\geq 1}$ and $(\alpha_{2,n})_{n\geq 1}$ strictly decreasing to 0, in every open ball in $\mathcal{H}$ with radius strictly larger then $(\alpha_{1,1}^2 + \alpha_{2,1}^2)^{1/2}$, there is $z \in \mathcal{H}$ satisfying*

$$\|T_1^n z\| \geq \alpha_{1,n} |\lambda_1|^n \text{ and } \|T_2^n z\| \geq \alpha_{2,n} |\lambda_2|^n \text{ for all } n \geq 1.$$

Corollary 1.2. [4, Corollary 3.2] *If $\sigma_a(T_1)\backslash\sigma_p(T_1)$ and $\sigma_a(T_2)\backslash\sigma_p(T_2)$ both have a nonempty intersection with the domain $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$, then there is a dense set of vectors $z \in \mathcal{H}$ such that both the orbits* $\mathrm{Orb}(T_1, z)$ *and* $\mathrm{Orb}(T_2, z)$ *tend strongly to infinity.*

With some additional changes of the proof, the results in Theorem 1.1 and Corollary 1.2 can be easily extended up to a *finite* sequence of operators $\{T_1, T_2, \ldots, T_k\}$ in $\mathcal{B}(\mathcal{H})$, and sequences $(a_{i,n})_{n\geq 1}$, $i = 1, 2, \ldots, k$ of positive numbers strictly decreasing to 0.

In general, for *infinite* sequence operators we need to make some additional restrictions upon the sequences $(a_{i,n})_{n\geq 1}$.

However, as in [4], our main tools are the following two results.

Proposition 1.3. [4, Prop. 2.1] *Let $T \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \sigma_a(T)\backslash\sigma_p(T)$.*

(*a*) *For every sequence of almost eigenvectors $(x_n)_{n\geq 1}$ for λ and every $h \in \mathcal{H}$, $\lim_{n\to\infty} \langle x_n | h \rangle = 0$.*

(*b*) *If E is any orthonormal basis for $\mathcal{H}$, then there is a sequence $(y_k)_{k\geq 1}$ of almost eigenvectors for λ, such that the sets $\{e \in E : \langle e|\, y_k\rangle \neq 0\}$, $k \geq 1$ are all finite and pairwise disjoint.*

Lemma 1.4. [4, Lemma 2.2] *If $A \in \mathcal{B}(\mathcal{H})$ and $(u_n)_{n\geq 1}$ is a sequence in $\mathcal{H}$ which tends weakly to 0, then for every $u \in H$ and every $\delta > 0$:*

(*a*) $\limsup_{n\to\infty} \|A(u + \delta u_n)\|^2 \geq \|Au\|^2$*; and*

(*b*) *if $\|Au_n\| \to \alpha$ as $n \to \infty$, then* $\lim_{n\to\infty} \|A(u + \delta u_n)\|^2 = \|Au\|^2 + \alpha^2\delta^2$.

Remark 1.5. The assertion (a) in Proposition 1.3, together with the Riesz's theorem for representation of bounded linear functional on Hilbert space, actually states that every sequence of almost eigenvectors for $\lambda \in \sigma_a(T)\backslash\sigma_p(T)$ tends weakly to 0.

Remark 1.6. If E is an orthonormal basis for $\mathcal{H}$, $(T_i)_{i\geq 1}$ is a sequence in $\mathcal{B}(\mathcal{H})$, $\lambda_i \in \sigma_a(T_i)\backslash\sigma_p(T_i)$, $i \geq 1$ and $(y_{i,k})_{k\geq 1}$ is a sequence of almost eigenvectors for λ_i as in Proposition 1.3.(b), then for each $i \geq 1$ we can find subsequence $(x_{i,k})_{k\geq 1}$ of $(y_{i,k})_{k\geq 1}$ such that the sets $E(x_{i,k}) = \{e \in E : \langle e|x_{i,k}\rangle \neq 0\}$, $i \geq 1$, $k \geq 1$ are pairwise disjoint. One way to do this is as follows.

Put $x_{1,1} = y_{1,1}$.

Suppose that, for some $l \geq 2$, we have found vectors $x_{i,j}$, $2 \leq i + j \leq l$, so that $E(x_{i,j})$, $2 \leq i + j \leq l$, are pairwise disjoint.

Now, since E is infinite (as an orthonormal basis of an infinite-dimensional Hilbert space) and $E(y_{i,j})$, $i \geq 1$, $j \geq 1$ are all finite, by induction we can find positive integers $N_i(j)$, $i + j = l + 1$ so that $E(x_{i,j})$, $2 \leq i + j \leq l$ and $E(y_{i,N_i(j)})$, $i + j = l + 1$, are pairwise disjoint. Put $x_{i,j} = y_{i,N_i(j)}$, for each i and j with $i + j = l + 1$.

2 Main result

Theorem 2.1. *Let $(T_i)_{i\geq 1}$ be a sequence in $\mathcal{B}(\mathcal{H})$ with $\sigma_a(T_i)\backslash\sigma_p(T_i) \neq \varnothing$ for all $i \geq 1$. Then for any sequence $(\lambda_i)_{i\geq 1}$ with $\lambda_i \in \sigma_a(T_i)\backslash\sigma_p(T_i)$, $i \geq 1$ and any family of strictly decreasing sequences of positive numbers $\{(a_{i,j})_{j\geq 1} : i = 1, 2, \dots\}$ with $a_{i,j} \to 0$ as $j \to \infty$ for all $i \geq 1$ and $\sum_{i\geq 1} a_{i,1}^2 < \infty$, in every open ball in $\mathcal{H}$ with radius $2(\sum_{i\geq 1} a_{i,1}^2)^{1/2}$ there is $z \in \mathcal{H}$ satisfying:*

$$\|T_i^n z\| \geq a_{i,n}\, |\lambda_i|^n\,, \ \textit{for all } i \geq 1 \textit{ and } n \geq 1.$$

Proof. By Proposition 1.3, Remark 1.5 and Remark 1.6, there are sequences $(x_{i,n})_{n\geq 1}$, $i = 1, 2, \ldots$, in $\mathcal{H}$ with the following properties:

(P.1) $\|x_{i,n}\| = 1$ for all $i \geq 1$ and $n \geq 1$;

(P.2) $\langle x_{i,n} \mid x_{j,m}\rangle = 0$ if $i \neq j$ or $n \neq m$;

(P.3) $(x_{i,n})_{n\geq 1}$ tends weakly to 0 for all $i \geq 1$; and

(P.4) $\left\|T_i^k x_{i,n}\right\| \to |\lambda_i|^k$ as $n \to \infty$, for all $i \geq 1$ and $k \geq 1$.

Fix $x \in \mathcal{H}$ and $0 < \varepsilon < 1/2$. Let

$$A_{i,j} = (a_{i,j}^2 - a_{i,j+1}^2)^{1/2},\ i \geq 1,\ j \geq 1.$$

We start with the sequence $(x + (1+\varepsilon)A_{1,1}x_{1,n})_{n\geq 1}$. By (P.3), (P.4) and Lemma 1.2.(b)

$$\lim_{n\to\infty} \|T_1(x + (1+\varepsilon)A_{1,1}x_{1,n})\|^2 = \|T_1 x\|^2 + (1+\varepsilon)^2 A_{1,1}^2 |\lambda_1|^2 > A_{1,1}^2 |\lambda_1|^2 .$$

Hence, there is a positive integer $N_1(1)$ so that

$$\left\|T_1(x + (1+\varepsilon)A_{1,1}x_{1,N_1(1)})\right\| > A_{1,1} |\lambda_1| .$$

Let $z_2 = x + (1+\varepsilon)A_{1,1}x_{1,N_1(1)}$.

Suppose that for some $l \geq 2$ we have found positive integers $N_i(j)$, $2 \leq i + j \leq l$ so that

$$N_i(j) < N_i(j+1) \text{ for all } i \text{ and } j\,, \tag{1}$$

and the vectors z_k, $2 \leq k \leq l$ defined with

$$z_k = x + (1+\varepsilon) \sum_{2\leq i+j\leq k} A_{i,j} x_{i,N_i(j)}, \tag{2}$$

satisfy

$$\left\|T_i^j z_k\right\| > (A_{i,j}^2 + \cdots + A_{i,k}^2)^{1/2} |\lambda_i|^j\,, \text{ for all } 2 \leq i + j \leq k. \tag{3}$$

Inductively we will find positive integers $N_s(l+1-s)$, $s = 1, 2, \ldots, l$ such that

$$N_s(l+1-s) > N_s(l-1),\ s = 1, 2, \ldots, l-1,$$

and the vectors $z_l^{(s)}$, $s = 1, 2, \dots, l$ defined with

$$z_l^{(s)} = z_l + (1+\varepsilon)\sum_{i=1}^{s} A_{i,l+1-i} x_{i,N_i(l+1-i)}$$

satisfy both (4) and (5) bellow,

$$\left\| T_i^j z_l^{(s)} \right\| > (A_{i,j}^2 + \cdots + A_{i,l}^2)^{1/2} \left|\lambda_i\right|^j, \text{ for all } 2 \le i + j \le l \tag{4}$$

and

$$\left\| T_i^{l+1-i} z_l^{(s)} \right\| > A_{i,l+1-i} \left|\lambda_i\right|^{l+1-i}, \text{ for all } 1 \le i \le s. \tag{5}$$

We begin with the sequence $(z_l + (1+\varepsilon)A_{1,l}x_{1,n})_{n\ge1}$. By Lemma 1.2.$(b)$ and (5)

$$\lim_{n\to\infty} \left\| T_1^l (z_l + (1+\varepsilon)A_{1,l}x_{1,n}) \right\|^2 = \left\| T_1^l z_l \right\|^2 + (1+\varepsilon)^2 A_{1,l}^2 \left|\lambda_1\right|^{2l} > A_{1,l}^2 \left|\lambda_1\right|^{2l}.$$

Hence, there is $n_0 > N_1(l-1)$ so that

$$\left\| T_1^l (z_l + (1+\varepsilon)A_{1,l}x_{1,n}) \right\| > A_{1,l} \left|\lambda_1\right|^l, \text{ for all } n \ge n_0.$$

On the other hand, by (3) for $k = l$, Lemma 1.2.(a) and (4)

$$\limsup_{n\to\infty} \left\| T_i^j (z_l + (1+\varepsilon)A_{1,l}x_{1,n}) \right\|^2 \ge \left\| T_i^j z_l \right\|^2 > (A_{i,j}^2 + \cdots + A_{i,l}^2)^{1/2} \left|\lambda_i\right|^{2j},$$

for all $2 \le i + j \le l$, which allows us to find $N_1(l) \ge n_0 > N_1(l-1)$ so that

$$z_l^{(1)} = z_l + (1+\varepsilon)A_{1,l}x_{1,N_1(l)}$$

satisfies $\left\| T_1^l z_l^{(1)} \right\| > A_{1,l} \left|\lambda_1\right|^l$ and

$$\left\| T_i^j z_l^{(1)} \right\| > (A_{i,j} + \cdots + A_{i,l})^{1/2} \alpha_{i,j} \left|\lambda_i\right|^j, \text{ for all } 2 \le i + j \le l.$$

Now, suppose that for some $1 \le s \le l-1$ we have found positive integers $N_1(l), N_2(l-1), \dots, N_s(l+1-s)$ with the desired properties.

We observe the sequence $(z_l^{(s)} + (1+\varepsilon)A_{s+1,l-s}x_{s+1,n})_{n\ge1}$. Applying again Lemma 1.2.(a) we have

$$\begin{aligned} \limsup_{n\to\infty} \left\| T_i^j (z_l^{(s)} + (1+\varepsilon)A_{s+1,l-s}x_{s+1,n}) \right\|^2 &\ge \left\| T_i^j z_l^{(s)} \right\|^2 \\ &> (A_{i,j}^2 + \cdots + A_{i,l}^2) \left|\lambda_i\right|^{2j}, \end{aligned}$$

for all i and j with:

(i) $1 \le i \le s$ and $2 \le i + j \le l + 1$; or

(ii) $s + 1 \le i \le l - 1$ and $2 \le i + j \le l$,

and, by Lemma 1.2.(b)

$$\lim_{n\to\infty} \left\| T_{s+1}^{l-s}(z_l^{(s)} + (1+\varepsilon)A_{s+1,l-s}x_{s+1,n}) \right\|$$
$$= \left\| T_{s+1}^{l-s} z_l^{(s)} \right\|^2 + (1+\varepsilon)^2 A_{s+1,l-s}^2 \left|\lambda_{s+1}\right|^{2(l-s)} > A_{s+1,l-s}^2 \left|\lambda_{s+1}\right|^{2(l-s)}.$$

Hence, we can find $N_{s+1}(l-s) > N_{s+1}(l-s-1)$ so that

$$z_l^{(s+1)} = z_l^{(s)} + (1+\varepsilon)A_{s+1,l-s}x_{s+1,N_{s+1}(l-s)}$$

satisfies (4) and (5) with $s + 1$ instead s.

Put $z_{l+1} = z_l^{(l)}$. Clearly

$$\left\| T_i^j z_{l+1} \right\| > (A_{i,j} + \ldots A_{i,l+1})^{1/2} \left|\lambda_i\right|^j, \text{ for all } 2 \le i + j \le l + 1.$$

By the previous discussion there are positive integers $N_i(j)$, $i \ge 1$, $j \ge 1$ such that (1) holds for all $i \ge 1$ and $j \ge 1$, and the sequence $(z_k)_{k\ge 2}$ given with (2) satisfies (3) for all $k \ge 2$. We will show that $(z_k)_{k\ge 2}$ is a Cauchy sequence in $\mathcal{H}$.

Let $\varepsilon' > 0$ be arbitrary. By the assumptions upon the sequences $(a_{i,j})_{j\ge 1}$, $i = 1, 2, \ldots$ there is a positive integer i_0 such that

$$\sum_{i\ge i_0+1} a_{i,1}^2 < \frac{\varepsilon'}{2(1+\varepsilon)^2}$$

and a positive integer $j_0 > i_0$ such that

$$\left|a_{i,j_1}^2 - a_{i,j_2}^2\right| < \frac{\varepsilon'}{2(i_0+1)(1+\varepsilon)^2}, \text{ for all } 1 \le i \le i_0,\ j_1 > j_0,\ j_2 > j_0.$$

Now, if $l > k > i_0 + j_0$, then by the definition of $A_{i,j}$, $i \ge 1, j \ge 1$ and the assumption that $(a_{i,j})_{j\geq 1}$, $i=1,2,\ldots$ are decreasing sequences we obtain

$$\begin{aligned}
\|z_l - z_k\|^2 &= (1+\varepsilon)^2 \sum_{k+1\leq i+j\leq l} A_{i,j}^2 \\
&< (1+\varepsilon)^2 \left[\sum_{i=1}^{i_0}(a_{i,k+1-i}^2 - a_{i,l+1-i}^2) + \sum_{i=i_0+1}^{l-1}(a_{i,1}^2 - a_{i,l+1-i}^2)\right] \\
&< (1+\varepsilon)^2 \left[\sum_{i=1}^{i_0}(a_{i,k+1-i}^2 - a_{i,l+1-i}^2) + \sum_{i=i_0+1}^{\infty} a_{i,1}^2\right] \\
&< (1+\varepsilon)^2 \left[\frac{\varepsilon' i_0}{2(i_0+1)(1+\varepsilon)^2} + \frac{\varepsilon'}{2(1+\varepsilon)^2}\right] \\
&< \varepsilon'.
\end{aligned}$$

Since $\mathcal{H}$ is a Hilbert space, there is $z\in\mathcal{H}$ such that $z=\lim_{k\to\infty} z_k$ (relative to the norm topology), and this vector is with the desired properties:

1. Keeping in mind that $(a_{i,j})_{j\geq 1}$, $i=1,2,\ldots$ are decreasing sequences we have

$$\begin{aligned}
\|x-z\|^2 &= \lim_{k\to\infty}\|x-z_k\|^2 = (1+\varepsilon)^2 \lim_{k\to\infty}\sum_{2\leq i+j\leq k} A_{i,j}^2 \\
&< 4\lim_{k\to\infty}\sum_{i=1}^{k-1}(a_{i,1}^2 - a_{i,k+1-i}^2) < 4\lim_{k\to\infty}\sum_{i=1}^{k-1} a_{i,1}^2 \\
&< 4\sum_{i=1}^{\infty} a_{i,1}^2.
\end{aligned}$$

   and consequently $\|x-z\| < 2\left(\sum_{i\geq 1} a_{i,1}^2\right)^{1/2}$,

2. For given positive integers i and n,

$$\|T_i^n z_k\| > (A_{i,n}^2 + \cdots + A_{i,k}^2)^{1/2}\,|\lambda_i|^n = (a_{i,n}^2 - a_{i,k+1}^2)^{1/2}\,|\lambda_i|^n\,,$$

   for all $k\geq n+i$, and consequently

$$\|T_i^n z\| = \lim_{k\to\infty}\|T_i^n z_k\| \geq a_{i,n}\,|\lambda_i|^n\,. \qquad \blacksquare$$

3 On orbits tending strongly to infinity

Theorem 2.1 allows us to give sufficient conditions under which, given a sequence $(T_i)_{i\geq 1}$ of operators in $\mathcal{B}(\mathcal{H})$ the space $\mathcal{H}$ will contain a dense set of vectors with orbits under each T_i tending strongly to infinity.

Corollary 3.1. *Let $(T_i)_{i\geq 1}$ be a sequence in $\mathcal{B}(\mathcal{H})$ such that for every $i \geq 1$ the set $\sigma_a(T_i)\backslash\sigma_p(T_i)$ has a nonempty intersection with $\{\lambda \in \mathbb{C} : |\lambda| > 1+\beta\}$ for some $\beta > 0$. Then there is a dense set of $z \in \mathcal{H}$ with $\operatorname{Orb}(T_i, z)$ tending strongly to infinity for every $i \geq 1$.*

Proof. Let $\lambda_i \in (\sigma_a(T_i)\backslash\sigma_p(T_i)) \cap \{\lambda \in \mathbb{C} : |\lambda| > 1+\beta\}$, $i \geq 1$. Let $1 < q < 1+\beta \leq \inf_{i\geq 1}|\lambda_i|$ and, for given $\delta > 0$, choose $0 < C < \delta q^2(q^2-1)/2$. If $x \in \mathcal{H}$ and

$$a_{i,j} = C^{\,1/2}q^{-(i+j)},\ i \geq 1,\ j \geq 1,$$

then, for $z \in \mathcal{H}$ from the proof of Theorem 2.1, $\|x - z\| < \delta$ and for each $i \geq 1$,

$$\|T_i^n z\| \geq a_{i,n}|\lambda_i|^n = C^{\,1/2}q^{-i}|\lambda_i q^{-1}|^n \to \infty \text{ as } n \to \infty. \qquad \blacksquare$$

Corollary 3.2. *Let $(T_i)_{i\geq 1}$ be a sequence of invertible operators in $\mathcal{B}(\mathcal{H})$ such that for each $i \geq 1$, the set $\sigma_a(T_i)\backslash\sigma_p(T_i)$ has a nonempty intersection with both $\{\lambda \in \mathbb{C} : |\lambda| > 1+\gamma\}$ and $\{\lambda \in \mathbb{C} : 0 < |\lambda| < \gamma\}$ for some $0 < \gamma < 1$. Then there is a dense set of $z \in \mathcal{H}$ for which both $\operatorname{Orb}(T_i, z)$ and $\operatorname{Orb}(T_i^{-1}, z)$ tend strongly to infinity for every $i \geq 1$.*

Proof. If $T \in \mathcal{B}(\mathcal{H})$ is invertible operator, then

$$\lambda \in \sigma_a(T^{-1}) \text{ if, and only if, } \lambda^{-1} \in \sigma_a(T), \tag{6}$$

and

$$\lambda \in \sigma_p(T^{-1}) \text{ if, and only if, } \lambda^{-1} \in \sigma_p(T). \tag{7}$$

By our assumption, (6) and (7), for each $i \geq 1$, both $\sigma_a(T_i)\backslash\sigma_p(T_i)$ and $\sigma_a(T_i^{-1})\backslash\sigma_p(T_i^{-1})$ will have a nonempty intersection $\{\lambda \in \mathbb{C} : |\lambda| > 1+\beta\}$ for $\beta = \min\{\gamma^{-1}-1, \gamma\}$. Now the assertion will follow from Corollary 3.1 applied on the sequence $\{T_1, T_1^{-1}, T_2, T_2^{-1}, \dots\}$. $\blacksquare$

Corollary 3.3. *Let $(T_i)_{i\geq 1}$ be a sequence in $\mathcal{B}(\mathcal{H})$ such that for every $i \geq 1$ the set $\sigma_a(T_i)\backslash\sigma_p(T_i)$ has a nonempty intersection with $\{\lambda \in \mathbb{C} : |\lambda| > 1+\beta\}$ for some $\beta > 0$ and $(T_i)_{i\geq 1}$ uniformly converges to $T \in \mathcal{B}(\mathcal{H})$. Then there is a vector $z \in \mathcal{H}$ with $\operatorname{Orb}(T, z)$ tending strongly to infinity.*

Proof. Let $\lambda_i \in (\sigma_a(T_i)\backslash\sigma_p(T_i)) \cap \{\lambda \in \mathbb{C} : |\lambda| > 1+\beta\}$, $i \geq 1$. By the uniform convergence of $(T_i)_{i\geq 1}$ to $T \in \mathcal{B}(\mathcal{H})$, there is $M > 1+\beta$ such that $\|T_i\| \leq M$ for all $i \geq 1$. Then, since $\sigma(T_i) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|T_i\|\}$, for each $i \geq 1$, the sequence $(\lambda_i)_{i\geq 1}$ is bounded and hence, there is a subsequence $(\lambda_{i_k})_{k\geq 1}$ and λ_0 in $\{\lambda \in \mathbb{C} : 1+\beta \leq |\lambda| \leq M\}$, so that $|\lambda_{i_k} - \lambda_0| \to 0$, as $k \to \infty$.

We will show that $\lambda_0 \in \sigma_a(T)$. If we assume that $\lambda_0 \notin \sigma_a(T)$, then we can find $c > 0$ (cf. [2, Proposition VII.6.4]) so that

$$\|Tx - \lambda_0 x\| \geq c\|x\|, \text{ for all } x \in \mathcal{H}.$$

Let k_0 is such that $\|T_{i_{k_0}} - T\| < c/4$ and $|\lambda_{i_{k_0}} - \lambda_0| < c/4$. Then

$$\begin{aligned} c\,\|x\| &\leq \|Tx - \lambda_0 x\| \\ &\leq \|Tx - T_{i_{k_0}}x\| + \|T_{i_{k_0}}x - \lambda_{i_{k_0}}x\| + \|\lambda_{i_{k_0}}x - \lambda_0 x\| \\ &< \frac{c}{4}\|x\| + \|T_{i_{k_0}}x - \lambda_{i_{k_0}}x\| + \frac{c}{4}\,\|x\|\,, \end{aligned}$$

for every $x \in \mathcal{H}$ and, consequently

$$\|T_{i_{k_0}}x - \lambda_{i_{k_0}}x\| \geq c\|x\|/2, \text{ for all } x \in \mathcal{H}.$$

But, then $\lambda_{i_{k_0}} \notin \sigma_a(T_{i_{k_0}})$, which contradicts the choice of $\lambda_{i_{k_0}}$. Hence $\lambda_0 \in \sigma_a(T)$. Now, since $|\lambda_0| \geq 1+\beta$ and $\lambda_0 \in \sigma_a(T)$, we have one of the following possibilities:

1. $\lambda_0 \in \sigma_p(T)$. Then for every $z \in \ker(T - \lambda_0)\backslash\{0\}$ the orbit $\mathrm{Orb}(T, z)$ will tend strongly to infinity.

2. $\lambda_0 \in \sigma_a(T)\backslash\sigma_p(T)$. Then there is a dense set of vectors in $\mathcal{H}$ with orbits under T tending strongly to infinity. Moreover, by Corollary 3.1, applied on the sequence $\{T, T_1, T_2, \dots\}$, there is a dense set of vectors $z \in \mathcal{H}$ with orbits under T and each T_i, $i \geq 1$, tending strongly to infinity. ■

If $(T_i)_{i\geq 1}$ is a sequence in $\mathcal{B}(\mathcal{H})$ satisfying the following, weaker condition then the one in Corollary 3.1,

$$(\sigma_a(T_i)\backslash\sigma_p(T_i)) \cap \{\lambda \in \mathbb{C} : |\lambda| > 1\} \neq \varnothing, \text{ for all } i \geq 1, \tag{8}$$

the space may still contain a dense set of vectors with orbits under each T_i, $i \geq 1$ tending strongly to infinity. As the next example shows, this may occur even in the case when $\|T_i\| \to 1$ as $i \to \infty$. However, if in addition $\|T_i - T\| \to 0$ as $i \to \infty$, the conclusion in Corollary 3.3 fails: $\|T\| = 1$, and hence $\mathrm{Orb}(T, z)$ remains in the ball $\{x \in \mathcal{H} : \|x\| \leq \|z\|\}$, for every $z \in \mathcal{H}$.

Example 3.4. Let S be the unilateral forward shift on $\ell^2(\mathbb{N})$:

$$Se_n = e_{n+1}, \ n = 1, 2, \dots \ ,$$

where $\{e_n : n \in \mathbb{N}\}$ is the standard orthonormal basis for $\ell^2(\mathbb{N})$. Given a sequence of positive numbers $(a_i)_{i\geq 1}$ so that $a_i > 1$ for all $i \geq 1$ and $a_i \to 1$ as $i \to \infty$ let

$$T_i = a_i S, \ \ i = 1, 2, \dots \ .$$

Then T_i is unilateral injective forward weighted shift (with weights all equal a_i) and hence (cf. [8, Theorem 6])

$$\sigma_p(T_i) = \varnothing \text{ and } \sigma_a(T_i) = \{\lambda \in \mathbb{C} : |\lambda| = a_i\}.$$

Obviously, $(T_i)_{i\geq 1}$ satisfies the weaker condition (8) and yet there is a dense set of vectors in $\ell^2(\mathbb{N})$ with orbits under each T_i tending strongly to infinity. Actually $\|T_i^n x\| = \|(a_i S)^n x\| = a_i^n \|x\| \to \infty$, as $n \to \infty$, for all $x \neq 0$ and $i \geq 1$. Moreover, $(T_i)_{i\geq 1}$ uniformly converges to S: since $\|S\| = 1$, by the choice of the sequence $(a_i)_{i\geq 1}$, $\|T_i - S\| = (a_i - 1)\|S\| = a_i - 1 \to 0$, as $i \to \infty$. But S does not have orbits tending strongly to infinity: $\|S^n x\| = \|x\|$ for all $x \in \ell^2(\mathbb{N})$.

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Address

Sonja Mančevska:

University of St. Kliment Ohridski - Bitola, Faculty of Technical Sciences,

Ivo Lola Ribar b.b., 7000 Bitola, Macedonia

E-mail: sonja.manchevska@uklo.edu.mk

Marija Orovčanec:

Ss. Cyril and Methodius University - Skopje
Faculty of Natural Sciences and Mathematics, Institute of Mathematics
Gazi Baba b.b., P.O.BOX 162, 1000 Skopje, Macedonia

E-mail: marijaor@iunona.pmf.ukim.edu.mk