Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **21:2** (2007), 173–184

ON UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS BY USING GENERALIZED SÂLÂGEAN OPERATOR

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Abstract

In this paper we have introduced a subclass $\mathcal{AR}(n,\xi,\alpha,\beta,\lambda)$ of univalent functions with negative coefficients defined by Salagean operator $\mathcal{D}^n$. We have obtained sharp results for coefficient estimates, distortion and closure bounds, Hadamard product and other results.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$. For a function $f(z)$ in $\mathcal{A}$, we define the following

$$\mathcal{D}^0 f(z) = f(z) \tag{1.2}$$

$$\mathcal{D}^1 f(z) = (1-\lambda)f(z) + \lambda z f'(z) = \mathcal{D}_\lambda f(z), \quad \lambda \geq 0 \tag{1.3}$$

$$\mathcal{D}^n f(z) = \mathcal{D}_\lambda(\mathcal{D}^{n-1} f(z)). \tag{1.4}$$

Also, from (1.3) and (1.4) we note that

$$\mathcal{D}^n f(z) = z + \sum_{k=2}^{\infty} (1+(k-1)\lambda)^n a_k z^k \tag{1.5}$$

2000 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Univalent functions, Starlike function, Sâlâgean operator, Integral operator, Distortion theorem, Hadamard product.
Received: June 8, 2007

if we put $\lambda = 1$, we have Sâlâgean operator [6].

A function $f(z)$ belonging to $\mathcal{A}$ is in the class $\mathcal{A}S(n, \xi, \alpha, \beta, \lambda)$ if and only if

$$\left| \frac{\frac{z(D^n f(z))'}{D^n f(z)} - 1}{2\xi\left(\frac{z(D^n f(z))'}{D^n f(z)} - \alpha\right) - \left(\frac{z(D^n f(z))'}{D^n f(z)} - 1\right)} \right| < \beta \tag{1.6}$$

where $0 \leq \alpha < \frac{1}{2\xi}, 0 < \beta \leq 1, \frac{1}{2} \leq \xi \leq 1, n \in I\!N \cup \{0\}, \quad z \in \mathcal{U}$.

Let T denote the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \ , \quad (a_k \geq 0). \tag{1.7}$$

Now we define the class $\mathcal{AR}(n, \xi, \alpha, \beta, \lambda)$ by

$$\mathcal{AR}(n, \xi, \alpha, \beta, \lambda) = \mathcal{AS}(n, \xi, \alpha, \beta, \lambda) \cap T.$$

We note that by specializing the parameters n, λ, ξ, α and λ we have the following subclasses

(i) the class $\mathcal{AR}(0, 1, 0, 1, 0)$ is precisely the class of starlike functions in $\mathcal{U}$.

(ii) the class $\mathcal{AR}(0, 1, \alpha, 1, 0)$ is the class of starlike functions of order $\alpha \ \ (0 \leq \alpha < 1)$.

(iii) the class $\mathcal{AR}(0, \frac{\alpha+1}{2}, 0, \beta, 0)$ is the class studied by Lakshminarsimhan [4].

(iv) the class $\mathcal{AR}(0, \xi, \alpha, \beta, 0)$ is the class studied by S. R. Kulkarni [3].

2 Coefficients estimates and other properties

Theorem 1. *Let f be defined by (1.7). Then $f \in \mathcal{AR}(n, \xi, \alpha, \beta, \lambda)$ if and only if*

$$\sum_{k=2}^{\infty} (1 + (k-1)\lambda)^n [(k-1)(1-\beta) + 2\beta\xi(k-\alpha)] a_k \leq 2\beta\xi(1-\alpha) \tag{2.1}$$

$n \in I\!N \cup \{0\}, 0 < \beta \leq 1, 0 \leq \alpha < \frac{1}{2\xi}, \frac{1}{2} \leq \xi \leq 1, \lambda \geq 0$.

Proof. For $|z| = 1$, we get

$$
\begin{aligned}
&|z(D^n f(z))' - D^n f(z)| - \beta|2\xi(z(D^n f(z))' - \alpha D^n f(z)) \\
&-(z(D^n f(z))' - D^n f(z))| = |-\sum_{n=2}^{\infty}(1+(k-1)\lambda)^n(k-1)a_k z^k| \\
&-\beta|2\xi(1-\alpha) - 2\xi\sum_{k=2}^{\infty}(k-\alpha)(1+(k-1)\lambda)^n a_k z^k \\
&+\sum_{k=2}^{\infty}(1+(k-1)\lambda)^n(k-1)a_k z^k| \\
&\leq \left[\sum_{k=2}^{\infty}(1+(k-1)\lambda)^n[(k-1)(1-\beta)+2\beta\xi(k-\alpha)]a_k - 2\beta\xi(1-\alpha)\right] \\
&\leq 0,
\end{aligned}
$$

by hypothesis. Thus by maximum modulus theorem, we have
$f \in \mathcal{AR}(n, \xi, \alpha, \beta, \lambda)$.

Conversely, suppose that $f \in \mathcal{AR}(n, \xi, \alpha, \beta, \lambda)$, therefore the condition (1.6) gives us

$$
\begin{aligned}
&\left|\frac{\frac{(z(D^n f(z))'}{D^n f(z)} - 1}{2\xi(\frac{z(D^n f(z))'}{D^n f(z)} - \alpha) - (\frac{z(D^n f(z))'}{D^n f(z)} - 1)}\right| = \\
&|[-\sum_{k=2}^{\infty}(1+(k-1)\lambda)^n(k-1)a_k z^{k-1}]/[2\xi(1-\alpha) - 2\xi \times \\
&\sum_{k=2}^{\infty}(k-\alpha)(1+(k-1)\lambda)^n a_k z^{k-1} \\
&+\sum^{\infty})_{k=2}(1+(k-1)\lambda)^n(k-1)a_k z^{k-1}]| < \beta.
\end{aligned}
$$

Since $|Re(z)| < |z|$ for all z, we obtain

$$
\begin{aligned}
&Re\{[\sum_{k=2}^{\infty}(1+k-1)\lambda)^n(k-1)a_k z^{k-1}]/[2\xi(1-\alpha) - 2\xi \times \\
&\sum_{k=2}^{\infty}(k-\alpha)(1+k-1)\lambda)^n a_k z^{k-1} + \sum_{k=2}^{\infty}(1+(k-1)\lambda)^n(k-1)a_k z^{k-1}]\} \\
&< \beta.
\end{aligned}
$$

Let $z \to 1^-$ through real values, so we have (2.1). The result is sharp for the function

$$f(z) = z - \frac{2\beta\xi(1-\alpha)}{((k-1)(1-\beta)+2\beta\xi(k-\alpha))(1+(k-1)\lambda)^n} z^k, \quad k \geq 2.$$

□

Corollary 2.1 : Let $f \in T$ belong to the class $\mathcal{AR}(n,\xi,\alpha,\beta,\lambda)$, then

$$a_k \leq \frac{2\beta\xi(1-\alpha)}{((k-1)(1-\beta)+2\beta\xi(k-\alpha))(1+(k-1)\lambda)^n}, \quad k \geq 2. \tag{2.2}$$

Theorem 2. *Let $f \in T$ belong to $\mathcal{AR}(n,\xi,\alpha,\beta,\lambda)$, then for $|z| \leq r < 1$, we have*

$$r - r^2 \frac{2\beta\xi(1-\alpha)}{(1+\lambda)^n((1-\beta)+2\beta\xi(2-\alpha))} \leq |D^n f(z)| \leq r + r^2 \frac{2\beta\xi(1-\alpha)}{(1+\lambda)^n((1-\beta)+2\beta\xi(2-\alpha))} \tag{2.3}$$

$$1 - 2r\frac{2\beta\xi(1-\alpha)}{(1+\lambda)^n((1-\beta)+2\beta\xi(2-\alpha))} \leq |(D^n f(z))'| \leq 1 + 2r\frac{2\beta\xi(1-\alpha)}{(1+\lambda)^n((1-\beta)+2\beta\xi(2-\alpha))}. \tag{2.4}$$

The above bounds are sharp.

Proof. By Theorem 2.1, we have

$$\sum_{k=2}^{\infty}(1+(k-1)\lambda)^n[(k-1)(1-\beta)+2\beta\xi(k-\alpha)]a_k \leq 2\beta\xi(1-\alpha),$$

then we get

$$(1+\lambda)^n((1-\beta)+2\beta\xi(2-\alpha))a_k \leq \sum_{k=2}^{\infty}(1+(k-1)\lambda)^n[(k-1)(1-\beta)+2\beta\xi(k-\alpha)]a_k \leq 2\beta\xi(1-\alpha),$$

then

$$\sum_{k=2}^{\infty} a_k \leq \frac{2\beta\xi(1-\alpha)}{(1+\lambda)^n((1-\beta)+2\beta\xi(1-\alpha))}.$$

Hence

$$\begin{aligned} |D^n f(z)| &\leq |z| + |z|^2 (1+\lambda)^n \sum_{k=2}^{\infty} a_k \\ &\leq r + r^2 (1+\lambda)^n \sum_{k=2}^{\infty} a_k \leq r + r^2 \frac{2\beta\xi(1-\alpha)}{((1-\beta)+2\beta\xi(2-\alpha))} \end{aligned}$$

and

$$\begin{aligned} |D^n f(z)| &\geq r - r^2 (1+\lambda)^n \sum_{k=2}^{\infty} a_k \\ &\geq r - r^2 \frac{2\beta\xi(1-\alpha)}{((1-\beta)+2\beta\xi(2-\alpha))}, \end{aligned}$$

thus (2.3) is true. Further

$$|(D^n f(z))'| \leq 1 + 2r(1+\lambda)^n \sum_{k=2}^{\infty} a_k \leq 1 + 2r \frac{2\beta\xi(1-\alpha)}{(1-\beta)+2\beta\xi(2-\alpha)}$$

and also

$$|D^n f(z))'| \geq 1 - 2r \frac{2\beta\xi(1-\alpha)}{(1-\beta)+(2\beta\xi(2-\alpha)}.$$

The result is sharp for the function $f(z)$, defined by

$$f(z) = z - \frac{2\beta\xi(1-\alpha)}{(1-\beta)+2\beta\xi(2-\alpha)} z^2, \quad z = \mp r.$$

This completes the proof of theorem. □

Theorem 3. *Let* $n \in I\!N \cup \{0\}, \lambda \geq 0, 0 < \beta \leq 1, 0 \leq \alpha_1 \leq \alpha_2 < \frac{1}{2\xi}$ *and* $\frac{1}{2} \leq \xi \leq 1$. *Then* $\mathcal{AR}(n,\xi,\alpha_2,\beta,\lambda) \subset \mathcal{AR}(n,\xi,\alpha_1,\beta,\lambda)$.

Proof. By assumption we have

$$\begin{aligned} &\frac{2\beta\xi(1-\alpha_2)}{(1+(k-1)\lambda)^n[(k-1)(1-\beta)+2\beta\xi(k-\alpha_2)]} \\ &\leq \frac{2\beta\xi(1-\alpha_1)}{(1+(k-1)\lambda)^n[(k-1)(1-\beta)+2\beta\xi(k-\alpha_1)]}. \end{aligned}$$

Thus, $f(z) \in \mathcal{AR}(n,\xi,\alpha_2,\beta,\lambda)$ implies that

$$\begin{aligned} \sum_{k=2}^{\infty} (1+(k-1)\lambda)^n a_k &\leq \frac{2\beta\xi(1-\alpha_2)}{(k-1)(1-\beta)+2\beta\xi(k-\alpha_2)} \\ &\leq \frac{2\beta\xi(1-\alpha_1)}{[(k-1)(1-\beta)+2\beta\xi(k-\alpha_1)]} \end{aligned}$$

then $f(z) \in \mathcal{AR}(n, \xi, \alpha_1, \beta, \lambda)$. □

Theorem 4. *The set $\mathcal{AR}(n, \xi, \alpha, \beta, \lambda)$ is the convex set.*

Proof. Let $f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i} z^k \quad (i = 1, 2)$ belong to $\mathcal{AR}(n, \xi, \alpha, \beta, \lambda)$ and let
$g(z) = \zeta_1 f_1(z) + \zeta_2 f_2(z)$, with ζ_1 and ζ_2 nonnegative and $\zeta_1 + \zeta_2 = 1$, we can write

$$g(z) = z - \sum_{k=2}^{\infty} (\zeta_1 a_{k,1} + \zeta_2 a_{k,2}) z^k.$$

It is sufficient to show that $g(z) \in \mathcal{AR}(n, \xi, \alpha, \beta, \lambda)$ that means

$$\sum_{k=2}^{\infty} (1 + (k-1)\lambda)^n [(k-1)(1-\beta) + 2\beta\xi(k-\alpha)][\zeta_1 a_{k,1} + \zeta_2 a_{k,2}]$$
$$= \zeta_1 \sum_{k=2}^{\infty} (1 + (k-1)\lambda)^n [(k-1)(1-\beta) + 2\xi(k-\alpha)] a_{k,1}$$
$$+ \zeta_2 \sum_{k=2}^{\infty} (1 + (k-1)\lambda)^n [(k-1)(1-\beta) + 2\beta\xi(k-\alpha)] a_{k,2}$$
$$\leq \zeta_1 (2\beta\xi(1-\alpha)) + \zeta_2 (2\beta\xi(1-\alpha)) = (\zeta_1 + \zeta_2)(2\beta\xi(1-\alpha)) = 2\beta\xi(1-\alpha).$$

Thus $g(z) \in \mathcal{AR}(n, \xi, \alpha, \beta, \lambda)$. □

We shall further try to obtain the extreme points in the following theorem.

Theorem 5. *Let $f_1(z) = z$ and*

$$f_k(z) = z - \frac{2\beta\xi(1-\alpha)}{(1 + (k-1)\lambda)^n ((k-1)(1-\beta) + 2\beta\xi(k-\alpha))} z^k$$

for all $k = 2, 3, \cdots; n \in I\!N \cup \{0\}; \lambda \geq 0; 0 < \beta \leq 1; 0 \leq \alpha < \frac{1}{2\xi}; \frac{1}{2} \leq \xi \leq 1$. Then $f(z)$ is in the subclass $\mathcal{AR}(n, \xi, \alpha, \beta, \lambda)$ if and only if it can be expressed in the form $f(z) = \sum_{k=1}^{\infty} \gamma_k z^k$ where ($\gamma_k \geq 0$ and $\sum_{k=1}^{\infty} \gamma_k = 1$ or $1 = \gamma_1 + \sum_{k=2}^{\infty} \gamma_k$).

Proof. Let $f(z) = \sum_{k=1}^{\infty} \gamma_k z^k$ where ($\gamma_k \geq 0$ and $\sum_{k=1}^{\infty} \gamma_k = 1$). Thus

$$f(z) = z - \sum_{k=2}^{\infty} \frac{2\beta\xi(1-\alpha)}{(1 + (k-1)\lambda)^n ((k-1)(1-\beta) + 2\beta\xi(k-\alpha))} \gamma_k z^k$$

and we obtain

$$\sum_{k=2}^{\infty}\left(\frac{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha))}{2\beta\xi(1-\alpha)}\right)\times$$
$$\gamma_k\frac{2\beta\xi(1-\alpha)}{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha))}=\sum_{k=2}^{\infty}\gamma_k=1-\gamma_1\leq 1.$$

In view of Theorem 2.1, this shows that $f(z)\in\mathcal{AR}(n,\xi,\alpha,\beta,\lambda)$. Conversely, suppose $f(z)$ of the form (1.7) belongs to $\mathcal{AR}(n,\xi,\alpha,\beta,\lambda)$ then

$$a_k\leq\frac{2\beta\xi(1-\alpha)}{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha))},\quad k\geq 2.$$

Putting $\gamma_k=\frac{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha))}{2\beta\xi(1-\alpha)}$, and $\gamma_1=1-\sum\limits_{k=2}^{\infty}\gamma_k$, then we have $f(z)=\gamma_1f_1(z)+\sum\limits_{k=2}^{\infty}\gamma_kf_k(z)$.

This completes the proof. □

3 Neighbourhood and Hadamard product properties

Definition 3.1 [5] : Let $\gamma\geq 0$ and $f(z)\in T$ of the form (1.7). The (k,γ)-neighbourhood of a function $f(z)$ defined by

$$N_{k,\gamma}(f)=\left\{g\in T:g(z)=z-\sum_{k=2}^{\infty}b_kz^k\ \text{ and }\ \sum_{k=2}^{\infty}k|a_k-b_k|\leq\gamma\right\},\quad (3.1)$$

for the identity function $e(z)=z$, we have

$$N_{k,\gamma}(e)=\left\{g\in T:g(z)=z-\sum_{k=2}^{\infty}b_kz^k\ \text{ and }\ \sum_{k=2}^{\infty}k|b_k|\leq\gamma\right\}.\qquad (3.2)$$

Theorem 6. *Let* $\gamma=\frac{4\beta\xi(1-\alpha)}{(1+\lambda)^n(1-\beta)+2\beta\xi(2-\alpha))}$. *Then* $\mathcal{AR}(n,\xi,\alpha,\beta,\lambda)\subset N_{k,\gamma}(e)$.

Proof. Let $f\in\mathcal{AR}(n,\xi,\alpha,\beta,\lambda)$, then we have

$$((1-\beta)+2\beta\xi(2-\alpha))(1+\lambda)^n\sum_{k=2}^{\infty}a_k$$
$$\leq\sum_{k=2}^{\infty}(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha))a_k\leq 2\beta\xi(1-\alpha),$$

therefore,

$$\sum_{k=2}^{\infty} a_k \leq \frac{2\beta\xi(1-\alpha)}{(1+\lambda)^n((1-\beta)+2\beta\xi(2-\alpha))}, \tag{3.3}$$

also we have for $|z| < r$

$$|f'(z)| \leq 1 + |z| \sum_{k=2}^{\infty} k a_k \leq 1 + r \sum_{k=2}^{\infty} k a_k.$$

In view of (3.3), we have

$$|f'(z)| \leq 1 + r \frac{2(2\beta\xi(1-\alpha))}{(1+\lambda)^n((1-\beta)+2\beta\xi(2-\alpha))}.$$

From above inequalities we get

$$\sum_{k=2}^{\infty} k a_k \leq \frac{4\beta\xi(1-\alpha)}{(1+\lambda)^n((1-\beta)+2\beta\xi(2-\alpha))} = \gamma,$$

therefore, $f \in N_{k,\gamma}(e)$. □

Definition 3.2 : The function $f(z)$ defined by (1.7) is said to be a member of the subclass $\mathcal{AR}(n, \xi, \alpha, \beta, \lambda, \zeta)$ if there exists a function $g \in \mathcal{AR}(n, \xi, \alpha, \beta, \lambda)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| \leq 1 - \zeta, \quad z \in \mathcal{U}, \quad 0 \leq \zeta < 1.$$

Theorem 7. *Let $g \in \mathcal{AR}(n, \xi, \alpha, \beta, \lambda)$ and*

$$\zeta = 1 - \frac{\gamma}{2} d(n, \alpha, \beta, \xi, \lambda). \tag{3.4}$$

Then $N_{k,\gamma}(g) \subset \mathcal{AR}(n, \xi, \alpha, \beta, \lambda, \zeta)$ where $n \in I\!N \cup \{0\}; \lambda \geq 0; 0 < \beta \leq 1$, $0 \leq \alpha < \frac{1}{2\xi}, \frac{1}{2} \leq \xi \leq 1, \lambda \geq 0, 0 \leq \zeta < 1$ and

$$d(n, \alpha, \beta, \xi, \lambda) = \frac{(1+\lambda)^n((1-\beta)+2\beta\xi(2-\alpha))}{(1+\lambda)^n((1-\beta)+2\beta\xi(2-\alpha)) - 2\beta\xi(1-\alpha)}.$$

Proof. Let $f \in N_{k,\gamma}(g)$, then by (3.3) we have $\sum\limits_{k=2}^{\infty} k|a_k - b_k| \leq \gamma$, then $\sum\limits_{k=2}^{\infty} |a_k - b_k| \leq \frac{\gamma}{2}$.

Since $g \in \mathcal{AR}(n,\xi,\alpha,\beta,\lambda)$, we have

$$\sum_{k=2}^{\infty} b_k \leq \frac{2\beta\xi(1-\alpha)}{(1+\lambda)^n((1-\beta)+2\beta\xi(2-\alpha))},$$

therefore,

$$\begin{aligned}\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\sum\limits_{k=2}^{\infty}|a_k-b_k|}{1-\sum\limits_{k=2}^{\infty} b_k} \\ &\leq \frac{\gamma}{2}\left(\frac{(1+\lambda)^n((1-\beta)+2\beta\xi(2-\alpha))}{(1+\lambda)^n((1-\beta)+2\beta\xi(2-\alpha))-2\beta\xi(1-\alpha)}\right) \\ &= \frac{\gamma}{2}d(n,\alpha,\beta,\xi,\lambda) = 1-\zeta.\end{aligned}$$

Then by Definition 3.2, we get $f \in \mathcal{AR}(n,\xi,\alpha,\beta,\lambda,\zeta)$. □

Theorem 8. *Let $f(z)$ and $g(z) \in \mathcal{AR}(n,\xi,\alpha_1,\beta,\lambda)$ be of the form (1.7) such that $f(z) = z - \sum\limits_{k=2}^{\infty} a_k z^k$ and $g(z) = z - \sum\limits_{k=2}^{\infty} b_k z^k$, where $a_k, b_k \geq 0$. Then the Hadamard product $h(z)$ defined by $h(z) = z - \sum\limits_{k=2}^{\infty} a_k b_k z^k$ is in the subclass $\mathcal{AR}(n,\xi,\alpha_2,\beta,\lambda)$ where*

$$\begin{aligned}\alpha_2 \quad \leq \quad & [((k-1)(1-\beta)+2\xi\beta(k-\alpha_1))^2(1+(k-1)\lambda)^n \\ & -2\beta\xi(1-\alpha_1)^2(k-1)(1-\beta)-(2\beta\xi)^2(1-\alpha_1)^2 k]/[((k-1)(1-\beta) \\ & +2\beta\xi(k-\alpha_1))^2(1+(k-1)\lambda)^n)-(2\xi\beta)^2(1-\alpha_1)^2].\end{aligned}$$

Proof. By Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \frac{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha_1))}{2\beta\xi(1-\alpha_1)} a_k \leq 1 \qquad (3.5)$$

and

$$\sum_{k=2}^{\infty} \frac{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha_1))}{2\beta\xi(1-\alpha_1)} b_k \leq 1. \qquad (3.6)$$

We have only to find the largest α_2 such that

$$\sum_{k=2}^{\infty} \frac{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha_2))}{2\beta\xi(1-\alpha_2)} a_k b_k \leq 1.$$

Now, by Cauchy-Schwarz inequality, we obtain

$$\sum_{k=2}^{\infty} \frac{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha_1))}{2\beta\xi(1-\alpha_1)} \sqrt{a_k b_k} \leq 1, \qquad (3.7)$$

we need only to show that

$$\frac{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha_2))}{2\beta\xi(1-\alpha_2)} a_k b_k \leq \frac{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha_1)}{2\beta\xi(1-\alpha_1)} \sqrt{a_k b_k}$$

equivalently

$$\sqrt{a_k b_k} \leq \frac{2\beta\xi(1-\alpha_2)}{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha_2))} \times \frac{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha_1))}{2\beta\xi(1-\alpha_1)}.$$

But from (3.7) we have

$$\sqrt{a_k b_k} \leq \frac{2\beta\xi(1-\alpha_1)}{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha_1))}.$$

Consequently, we need to prove that

$$\frac{2\beta\xi(1-\alpha_1)}{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha_1))} \leq \frac{(1-\alpha_2)((k-1)(1-\beta)+2\beta\xi(k-\alpha_1))}{(1-\alpha_1)((k-1)(1-\beta)+2\beta\xi(k-\alpha_2))}$$

or, equivalently, that

$$\alpha_2 \leq [-2\beta\xi(1-\alpha_1)^2(k-1)(1-\beta)+((k-1)(1-\beta)+2\xi\beta(k-\alpha_1))^2 \times (1+(k-1)\lambda)^n - (2\xi\beta)^2(1-\alpha_1)^2 k]/[((k-1)(1-\beta)+2\beta\xi(k-\alpha_1))^2 \times (1+(k-1)\lambda)^n) - (2\xi\beta)^2(1-\alpha_1)^2].$$

□

Theorem 9. *Let $f(z) \in \mathcal{AR}(n,\xi,\alpha,\beta,\lambda)$ be defined by (1.7) and c any real number with $c > -1$ then the function $G(z)$ defined as $G(z) = \frac{c+1}{z^c}\int_0^z s^{c-1} f(s)ds, \quad c > -1$, also belongs to $\mathcal{AR}(n,\xi,\alpha,\beta,\lambda)$.*

Proof. By virtue of $G(z)$ it follows from (1.7) that

$$G(z) = \frac{c+1}{z^c}\int_0^z \left(s^c - \sum_{k=2}^{\infty} a_k s^{k+c-1}\right) ds = z - \sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right) a_k z^k.$$

But

$$\sum_{k=2}^{\infty} \frac{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha))}{2\beta\xi(1-\alpha)}\left(\frac{c+1}{c+k}\right) a_k \leq 1,$$

since $\frac{c+1}{c+k} \leq 1$ and by Theorem 2.1, so the proof is complete. □

Theorem 10. *Let $f(z) \in \mathcal{AR}(n,\xi,\alpha,\beta,\lambda)$ be defined by (1.7) and $F_\mu(z) = (1-\mu)z + \mu\int_0^z \frac{f(s)}{s}ds$ $(\mu \geq 0, z \in \mathcal{U})$. Then $F_\mu(z)$ is also in $\mathcal{AR}(n,\xi,\alpha,\beta,\lambda)$ if $0 \leq \mu \leq 2$.*

Proof. Let f defined by (1.7) then

$$F_\mu(z) = (1-\mu)z + \lambda\int_0^z \left(\frac{s - \sum\limits_{k=2}^{\infty} a_k s^k}{s}\right) ds = z - \sum_{k=2}^{\infty}\frac{\mu}{k} a_k z^k.$$

By Theorem 2.1 and since $(\frac{\mu}{2} \leq 1)$ we have

$$\sum_{k=2}^{\infty} \frac{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha))}{2\beta\xi(1-\alpha)}(\frac{\mu}{k})a_k$$
$$\leq \sum_{k=2}^{\infty} \frac{(1+(k-1)\lambda)^n((k-1)(1-\beta)+2\beta\xi(k-\alpha))}{2\beta\xi(1-\alpha)}(\frac{\mu}{2})a_k \leq 1,$$

then $F_\mu(z)$ is in $\mathcal{AR}(n,\xi,\alpha,\beta,\lambda)$. □

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