Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **21:2** (2007), 185–197

ON NONLINEAR DISCONTINUOUS TWO-POINT BOUNDARY VALUE PROBLEMS FOR THIRD ORDER DIFFERENTIAL EQUATIONS

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Abstract

In this paper we prove existence of weak extremal solutions for third order nonlinear discontinuous two-point boundary value problems. Further, we obtain two weak differential inequalities for proving boundedness and uniqueness of solutions of related boundary value problems.

1 Introduction

The importance of boundary value problems in the theory of differential equations and their applications to different areas of science and technology are well known. This paper is concerned with proving existence of weak maximal and minimal solutions of a class of nonlinear discontinuous two-point boundary value problems of the form

$$y''' = f(t, y, y', y'') \quad a.e., \quad t \in I = [a, b] \tag{1.1}$$

$$y'(a) = y''(a) = y(b) = 0, \tag{1.2}$$

where $f : I \times R \times R \times R \to R$ is a function satisfying the following conditions;
(i) f is bounded on $I \times R \times R \times R$, i.e., there exists a constant $M > 0$ such that
$|f(t, x, y, z)| \leq M$, $\forall (t, x, y, z) \in I \times R \times R \times R$.
(ii) $f(t, x, y, z)$ is nondecreasing in x, y, and z for all most all $t \in I$.

2000 *Mathematics Subject Classification.* 34A10, 34B15.
Key words and phrases. Existence, uniqueness, maximal (minimal) weak solutions, differential inequalities, two-point boundary value problems.
Received: February 4, 2007

(iii) $f(.,x(.),y(.),z(.))$ is Lebesgue measurable for all Lebesgue measurable functions x, y and z on I.

Many authors [2, 4, 5, 6] have studied existence and uniqueness theorems, extremal solutions with discontinuous right hand side for second order nonlinear boundary value problems under certain generalized measurability and Lipschitz condition and also with monotonicity conditions. Recently Dhage [4] obtained existence of extremal solutions for second order nonlinear discontinuous boundary value problem under certain monotonicity conditions. Dhage also obtained weak differential inequalities which are applied to establish boundedness and uniqueness of solutions of related boundary value problems.

In this paper we obtain existence of maximal and minimal weak solutions for third order differential equations. Further weak differential inequalities are also obtained, which are useful for proving uniqueness and boundedness of related boundary value problems. This paper generalizes the results of Dhage [4] to third order two-point boundary value problems.

2 Existence of Weak Maximal and Minimal Solutions

In this section we prove existence of weak maximal and minimal solutions for the third order nonlinear differential equation (1.1) satisfying two-point boundary conditions (1.2).

Definition 2.1 The weak solution of a problem (1.1) satisfying (1.2) is a function $y \in H^2(I)$, satisfying the equations (1.1) and (1.2), where $H^2(I)$ denote the Sobolev space of all real valued functions on I, defined by

$$H^2(I) = \left\{ y \in AC(I,R)/y, y' \in L^1(I,R) \right\}, \tag{2.1}$$

where $AC(I,R)$ denote the space of all absolutely continuous functions mapping from I to R.

Let $||.||_{H^2}$ denote the usual norm in the Sobolev space $H^2(I)$ given by

$$||y||_{H^2} = \int_a^b |y(t)|dt + \int_a^b |y'(t)|dt + \int_a^b |y''(t)|dt. \tag{2.2}$$

It is well known that $H^2(I)$ is a Banach space with the above norm.

Definition 2.2 A partial ordering $\leq$ in Sobolev space $H^2(I)$ is given by $y \leq z$ if and only if $y(t) \leq z(t)$, $y'(t) \leq z'(t)$, and $y''(t) \leq z''(t)$ for all $t \in I$, and we write $y \leq z$ on I.

Lemma 2.1 $(H^2(I), \leq)$ *is a complete lattice.*

Proof. Let $u, v \in H^2(I)$ be such that $u \leq v$ on I. Then $u(t) \leq v(t)$, $u'(t) \leq v'(t)$, and $u''(t) \leq v''(t)$ for all $t \in I$. For any $w \in H^2(I)$, we have $(u+w)(t) \leq (v+w)(t)$, $(u+w)'(t) \leq (v+w)'(t)$, and $(u+w)''(t) \leq (v+w)''(t)$, for all $t \in I$, which implies that $u + w \leq v + w$ on I. Similarly if $\lambda \geq 0$, then $\lambda u \leq \lambda v$ on I. Therefore $H^2(I)$ is a vector lattice.

Let $u, v \in H^2(I)$ be such that $u \leq 0$, $v \leq 0$, and $u \leq v$ on I, then we have

$$\begin{aligned} ||u||_{H^2} &= \int_a^b |u(t)|dt + \int_a^b |u'(t)|dt + \int_a^b |u''(t)|dt \\ &\leq \int_a^b |v(t)|dt + \int_a^b |v'(t)|dt + \int_a^b |v''(t)|dt \\ &= ||v||_{H^2}. \end{aligned} \tag{2.3}$$

This shows that $(H^2(I), \leq)$ is a Banach lattice.
For $u, v \in H^2(I)$ with $u \leq 0$ and $v \leq 0$ on I, consider

$$\begin{aligned} ||u+v||_{H^2} &= \int_a^b |u(t)+v(t)|dt + \int_a^b |u'(t)+v'(t)|dt + \int_a^b |u''(t)+v''(t)|dt \\ &\leq \int_a^b |u(t)|dt + \int_a^b |u'(t)|dt + \int_a^b |u''(t)|dt \\ &\quad + \int_a^b |v(t)|dt + \int_a^b |v'(t)|dt + \int_a^b |v''(t)|dt \\ &= ||u||_{H^2} + ||v||_{H^2}. \end{aligned} \tag{2.4}$$

Hence $(H^2(I), \leq)$ is a complete lattice. $\square$

Definition 2.3 Let $S \subset H^2(I)$. A mapping $T : S \to H^2(I)$ is said to be isotone increasing if $u, v \in H^2(I)$, $u \leq v$ on I, then $Tu \leq Tv$ on I.

Now we state the following fixed point theorem of Tarski.

Theorem 2.1. (Tarski fixed point theorem [8]) *Let*

(i) $\Omega = (A, \leq)$ *be a complete lattice,*

(ii) f *be an increasing function on* A *to* A*,*

(iii) P *be the set of all fixed points of* f*.*

Then the set P *is non-empty and the system* $(P, \leq)$ *is a complete lattice; in particular* $\bigcup P = \bigcup E_x[f(x) \geq x] \in P$ *and* $\bigcap P = \bigcap E_x[f(x) \leq x] \in P$*.*

Now we prove the theorem on existence of extremal solutions for the problem (1.1) satisfying (1.2) by using the Tarski fixed point theorem.

Theorem 2.2. *Assume (i)-(iii) holds. Then the boundary value problem (1.1) satisfying (1.2) has weak maximal and weak minimal solutions on* I*.*

Proof. Consider a uniform bounded subset of the Sobolev space $H^2(I)$ by

$$S = \left\{ u \in H^2(I) / ||u||_{H^2} \leq N \right\}, \tag{2.5}$$

where $N = \frac{Mh^2}{6}\left(h^2 + 3h + 6\right)$ and $h = b - a$.
Clearly S is a nonempty, closed, convex and bounded subset of the complete lattice $H^2(I)$, so it is a complete lattice [3].

If $y(t)$ is a solution of the discontinuous boundary value problem (1.1) satisfying (1.2) if and only if it is a solution of the integral equation

$$y(t) = \int_a^b G(t,s) f(s, y(s), y'(s), y''(s)) ds, \quad t \in I \tag{2.6}$$

where $G(t,s)$ is a Green's function for the homogeneous boundary value problem

$$y'''(t) = 0 \tag{2.7}$$

satisfying

$$y'(a) = y''(a) = y(b) = 0, \tag{2.8}$$

given by

$$G(t,s) = \begin{cases} \frac{(t-s)^2 - (b-s)^2}{2}, & if \quad a \leq s \leq t \leq b \\ \frac{-(b-s)^2}{2}, & if \quad a \leq t \leq s \leq b. \end{cases}$$

Consider

$$\max_{t\in I}\int\limits_a^b |G(t,s)|ds = \max_{t\in I}\left\{\int\limits_a^t |G(t,s)|ds + \int\limits_t^b |G(t,s)|ds\right\}$$
$$= \max_{t\in I}\left\{\frac{(b-a)^3-(t-a)^3}{6}\right\}$$

The maximum value of the above function attains at $t = a$ and hence

$$\max_{t\in I}\int\limits_a^b |G(t,s)|ds \le \frac{(b-a)^3}{6}. \tag{2.9}$$

Again consider

$$\max_{t\in I}\int\limits_a^b |G_t(t,s)|ds = \max_{t\in I}\left\{\frac{(t-a)^2}{2}\right\} \le \frac{(b-a)^2}{2}. \tag{2.10}$$

Also consider

$$\max_{t\in I}\int\limits_a^b |G_{tt}(t,s)|ds = \max_{t\in I}\{t-a\} \le b-a. \tag{2.11}$$

Define the operator $T : S \to H^2(I)$ by

$$[Ty](t) = \int\limits_a^b G(t,s)f(s,y(s),y'(s),y''(s))ds, \forall\ t\in I. \tag{2.12}$$

Therefore the problem of existence of weak solutions of boundary value problem (1.1) satisfying (1.2) is equivalent to finding the fixed point of the operator T on S.

Claim. $T : S \to S$.

¿From the definition of $[Ty]$, it is absolutely continuous function on I. i.e. $[Ty] \in AC(I,R)$ for each $y \in S$. Since f satisfies (i) and (iii), implies that $f(.,y(.),y'(.),y''(.))$ is Lebesgue measurable on I, so $[Ty]'$, $[Ty]'' \in L^1(I,R)$ for all $y \in S$. Thus $T : S \to H^2(I)$.

Let $y \in S$, then

$$\begin{aligned}
||Ty||_{H^2} &= \int_a^b |[Ty](t)|dt + \int_a^b |[Ty]'(t)|dt + \int_a^b |[Ty]''(t)|dt \\
&\leq \int_a^b \left[\int_a^b |G(t,s)||f(s,y(s),y'(s),y''(s))|ds \right] dt \\
&\quad + \int_a^b \left[\int_a^b |G_t(t,s)||f(s,y(s),y'(s),y''(s))|ds \right] dt \\
&\quad + \int_a^b \left[\int_a^b |G_{tt}(t,s)||f(s,y(s),y'(s),y''(s))|ds \right] dt \\
&\leq \int_a^b M\frac{(b-a)^3}{6}dt + \int_a^b M\frac{(b-a)^2}{2}dt + \int_a^b M(b-a)dt \\
&= M\left[\frac{h^4}{6} + \frac{h^3}{2} + h^2\right] = N.
\end{aligned}$$

Hence the claim.

Let $y, z \in S$ be such that $y \leq z$ on I. Since f satisfies (ii), it follows that

$$\begin{aligned}
[Ty](t) &= \int_a^b G(t,s)f(s,y(s),y'(s),y''(s))ds \\
&\leq \int_a^b G(t,s)f(s,z(s),z'(s),z''(s))ds = [Tz](t),
\end{aligned}$$

$$\begin{aligned}
[Ty]'(t) &= \int_a^b G_t(t,s)f(s,y(s),y'(s),y''(s))ds \\
&\leq \int_a^b G_t(t,s)f(s,z(s),z'(s),z''(s))ds = [Tz]'(t),
\end{aligned}$$

and

$$[Ty]''(t) = \int_a^b G_{tt}(t,s)f(s,y(s),y'(s),y''(s))ds$$
$$\leq \int_a^b G_{tt}(t,s)f(s,z(s),z'(s),z''(s))ds = [Tz]''(t)$$

for all $t \in I$. Hence $Ty \leq Tz$ on I, which shows that T is isotone increasing on S. From Tarski fixed point theorem, the operator T has a fixed point, which is a solution of the boundary value problem (1.1) satisfying (1.2), and also the set of all solutions is a complete lattice. Hence the boundary value problem (1.1) satisfying (1.2) has weak maximal and weak minimal solutions on I. □

3 Weak Differential Inequalities And Applications

In this section we obtain two basic weak differential inequalities in terms of the weak extremal solutions of the boundary value problem (1.1) satisfying (1.2). Further, we apply the inequalities for proving boundedness and uniqueness of solutions of the related boundary value problem on I.

Theorem 3.1. *Assume (i)-(iii) holds. Further, if there is a function $w \in S$, where S is defined by (2.5) such that*

$$w''' \leq f(t,w,w',w'') \quad a.e., \quad t \in I \tag{3.1}$$

satisfying

$$w'(a) = w''(a) = w(b) = 0, \tag{3.2}$$

$$w'(t) \leq \int_a^t (t-s)f(s,w(s),w'(s),w''(s))ds \quad a.e., \quad t \in I \tag{3.3}$$

and

$$w''(t) \leq \int_a^t f(s,w(s),w'(s),w''(s))ds \quad a.e., \quad t \in I. \tag{3.4}$$

Then, there is a maximal weak solution y_M of the boundary value problem (1.1) satisfying (1.2) such that

$$w \leq y_M \quad on \quad I. \tag{3.5}$$

Proof. Let $\gamma = \sup S$. Consider the lattice interval $[w, \gamma]$, clearly this is a complete lattice. Now define the operator T on $[w, \gamma]$ as in (2.12).

First, we show that $T : [w, \gamma] \to [w, \gamma]$. For this, it suffices to show that if $y \in S$ is any element such that $w \leq y$ implies $w \leq Ty$ on I. From inequalities (3.1), (3.3), and (3.4), we have

$$\begin{aligned} w(t) &\leq \int_a^b G(t,s) f(s, w(s), w'(s), w''(s)) ds \\ &\leq \int_a^b G(t,s) f(s, y(s), y'(s), y''(s)) ds = [Ty](t), \end{aligned}$$

$$\begin{aligned} w'(t) &\leq \int_a^t (t-s) f(s, w(s), w'(s), w''(s)) ds \\ &= \int_a^b G_t(t,s) f(s, w(s), w'(s), w''(s)) ds \\ &\leq \int_a^b G_t(t,s) f(s, y(s), y'(s), y''(s)) ds = [Ty]'(t), \end{aligned}$$

and

$$\begin{aligned} w''(t) &\leq \int_a^t f(s, w(s), w'(s), w''(s)) ds \\ &= \int_a^b G_{tt}(t,s) f(s, w(s), w'(s), w''(s)) ds \\ &\leq \int_a^b G_{tt}(t,s) f(s, y(s), y'(s), y''(s)) ds = [Ty]''(t), \end{aligned}$$

for all $t \in I$. It follows that $w \leq Ty$ on I. Again as in the proof of Theorem 2.1, it is easily seen that T is isotone increasing on $[w, \gamma]$, and an application of Tarski fixed point theorem yields that there is a maximal weak solution y_M of the problem (1.1) satisfying (1.2) in $[w, \gamma]$. Hence we have

$$w \leq y_M \quad on \quad I.$$

□

Theorem 3.2. *Assume (i)-(iii) holds. Further, if there is a function $u \in S$, where S is defined by (2.5) such that*

$$u''' \geq f(t, u, u', u'') \quad a.e., \quad t \in I \tag{3.6}$$

satisfying

$$u'(a) = u''(a) = u(b) = 0, \tag{3.7}$$

$$u'(t) \geq \int_a^t (t-s) f(s, u(s), u'(s), u''(s)) ds \quad a.e., \quad t \in I \tag{3.8}$$

and

$$u''(t) \geq \int_a^t f(s, u(s), u'(s), u''(s)) ds \quad a.e., \quad t \in I. \tag{3.9}$$

Then, there is a minimal weak solution y_m of the boundary value problem (1.1) satisfying (1.2) such that

$$y_m \leq u \quad on \quad I. \tag{3.10}$$

Proof. The proof is similar to the proof of Theorem 3.1. □

Now we obtain boundedness and uniqueness of the weak solution of the boundary value problem (1.1) satisfying (1.2) on I.

Consider the problem

$$\phi''' = \psi(t, \phi) \quad , \quad t \in I \tag{3.11}$$

satisfying the two-point boundary conditions

$$\phi'(a) = \phi''(a) = \phi(b) = 0, \tag{3.12}$$

where $\phi : I \to R^+$, and $\psi : I \times R^+ \to R^+$ are functions.

Theorem 3.3. *Suppose that ψ satisfies (i)-(iii). Further, if the functions f and ψ satisfy the condition*

$$|f(t, y, z, w)| \leq \psi(t, |y|) \quad a.e., \quad t \in I \tag{3.13}$$

for all $y, z, w \in R$, then there is a maximal weak solution ϕ_M of the boundary value problem (3.11) satisfying (3.12) such that

$$|y| \leq \phi_M \quad on \quad I,$$

where y is any solution of the boundary value problem (1.1) satisfying (1.2) on I.

Proof. Let y be any solution of the boundary value problem (1.1) satisfying (1.2) on I. Then it is a solution of the integral equation

$$y(t) = \int_a^b G(t,s) f(s, y(s), y'(s), y''(s)) ds.$$

From (3.13) we have

$$|y(t)| \leq \int_a^b |G(t,s)||f(s, y(s), y'(s), y''(s))| ds$$

$$\leq \int_a^b |G(t,s)| \psi(s, |y(s)|) ds. \tag{3.14}$$

Therefore $|y(t)|$ is a solution of the problem

$$\phi''' \leq \psi(t, \phi) \qquad a.e., \quad t \in I \tag{3.15}$$

satisfying (3.12). If $y(t) \neq 0$, then

$$(|y(t)|)' \leq |y'(t)|, \quad and \quad (|y(t)|)'' \leq |y''(t)|, \quad t \in I.$$

Therefore

$$(|y(t)|)' \leq \int_a^b |G_t(t,s)||f(s, y(s), y'(s), y''(s))| ds$$

$$\leq \int_a^b |G_t(t,s)| \psi(t, |y(s)|) ds$$

$$= \int_a^t (t-s) \psi(t, |y(s)|) ds, \tag{3.16}$$

and

$$(|y(t)|)'' \leq \int_a^b |G_{tt}(t,s)||f(s, y(s), y'(s), y''(s))| ds$$

$$\leq \int_a^b |G_{tt}(t,s)| \psi(t, |y(s)|) ds$$

$$= \int_a^t (\psi(t, |y(s)|)ds. \tag{3.17}$$

From (3.15)-(3.17), and by an application of Theorem 3.1 yields that there exists a maximal solution ϕ_M of the boundary value problem (3.11) satisfying (3.12) such that

$$|y| \leq \phi_M \quad on \quad I.$$

□

Theorem 3.4. *Suppose that ψ satisfies (i)-(iii). Further, if the functions f and ψ satisfy the condition*

$$|f(t, y_1, y_2, y_3) - f(t, z_1, z_2, z_3)| \leq \psi(t, |y_1 - z_1|) \quad a.e., \quad t \in I \tag{3.18}$$

for all y_1, y_2, y_3, z_1, z_2, and $z_3 \in R$. Further, if the identically zero function is the only weak solution of the boundary value problem (3.11) satisfying (3.12) existing on I, then the boundary value problem (1.1) satisfying (1.2) has a unique solution on I.

Proof. Suppose the boundary value problem (1.1) satisfying (1.2) has two solutions y and z on I. Then we have

$$|y(t) - z(t)| \leq \int_a^b |G(t,s)||f(s, y(s), y'(s), y''(s)) - f(s, z(s), z'(s), z''(s))|ds$$

$$\leq \int_a^b |G(t,s)|\psi(s, |y(s) - z(s)|)ds. \tag{3.19}$$

Hence $|y(t) - z(t)|$ is a solution of (3.15). Again

$$(|y(t) - z(t)|)' \leq |y'(t) - z'(t)|$$

$$\leq \int_a^b |G_t(t,s)|\psi(s, |y(s) - z(s)|)ds$$

$$= \int_a^t (t - s)\psi(s, |y(s) - z(s)|)ds, \tag{3.20}$$

and

$$(|y(t)-z(t)|)'' \leq |y''(t)-z''(t)|$$

$$\leq \int_a^b |G_{tt}(t,s)|\psi(s,|y(s)-z(s)|)ds$$

$$= \int_a^t \psi(s,|y(s)-z(s)|)ds. \tag{3.21}$$

From (3.20), (3.21), and Theorem 3.1, we have

$$|y(t)-z(t)| \leq 0 \quad on \quad I.$$

Hence $y(t) = z(t)$, $\forall\ t \in I$. □

Example 3.1 Consider the boundary value problem

$$y''' = p(t)yq(y')r(y'') \quad a.e., \quad t \in [0,1] \tag{3.22}$$

satisfying

$$y'(0) = y''(0) = y(1) = 0, \tag{3.23}$$

where the functions $p, q, r : R \to R^+$ are given by

$$p(t) = \begin{cases} 1, & t \text{ is irrational} \\ 2, & t \text{ is rational} \end{cases}, q(y) = \begin{cases} 2, & y > 0 \\ 0, & y \leq 0 \end{cases}, \text{and } r(y) = \begin{cases} 1, & y > 0 \\ 0, & y \leq 0 \end{cases}.$$

It is easily seen that $f : [0,1] \times R \times R \times R \to R$ defined by

$$f(t,x,y,z) = p(t)xq(y)r(z), \quad \forall (t,x,y,z) \in [0,1] \times R \times R \times R$$

satisfies the conditions (i)-(iii). By the application of Theorem 2.1, the boundary value problem (3.22) satisfying (3.23) has weak maximal and minimal solutions on [0, 1].

Acknowledgement. The authors are grateful to the referee for his valuable suggestions.

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