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A COMMON FIXED POINT THEOREM USING IMPLICIT RELATION AND PROPERTY (E.A) IN METRIC SPACES

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Abstract

In this paper, we prove a common fixed point theorem for a quadruple of mappings by using an implicit relation [6] and property (E.A) [1] under weak compatibility.Our theorem improves and generalizes the main Theorems of Popa [6] and Aamri and Moutawakil [1].Various examples verify the importance of weak compatibility and property (E.A) in the existence of common fixed point and examples are also given to the implicit relation and to validate our main Theorem. We also show that property (E.A) and Meir-Keeler type contractive condition are independent to each other.

1. Introduction

The concept of weakly commuting mappings of Sessa [7] is sharpened by Jungck [3] and further generalized by Jungck and Rhoades [4]. Similarly, noncompatible mapping is generalized by Aamri and Moutawakil [1] called property (E.A). Noncompatibility is also important to study the fixed point theory. There may be pairs of mappings which are noncompatible but weakly compatible (see Example 1 of Popa [6] p. 34, and Example 2.1below).

Let A and S be two self-maps of a metric space (X, d). Mappings A and S are said to be *weakly commuting*[7] if

$$d(SAx, ASx) \leq d(Ax, Sx), \text{ for all } x \in X, \tag{1.1}$$

compatible [3] if

$$lim_{n\to\infty} d(ASx_n, SAx_n) = 0, \tag{1.2}$$

whenever there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

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noncompatible if there exists a sequence$\{x_n\}$ in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = t$, for some $t \in X$ and

$$lim_{n\to\infty} d(ASx_n, SAx_n) \text{ is either nonzero or nonexistent,} \tag{1.3}$$

and *weakly compatible* if they commute at their coincidence points, i.e.,

$$ASu = SAu \quad \text{whenever} \quad Au = Su, \quad \text{for some } u \in X. \tag{1.4}$$

2. Preliminaries

Property (E.A) [1]. Let A and S be two self-maps of a metric space (X, d) then they are said to satisfy *property (E.A)*, if there exists a sequence $\{x_n\}$ in X such that

$$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = t, \text{ for some } t \in X. \tag{2.1}$$

Notice that weakly compatible and property (E.A) are independent to each other:

Example 2.1. Consider $X = [0, 1]$ equipped with the usual metric d. Define $f, g : X \to X$ by:

$$f(x) = 1 - x, \text{ if } x \in [0, \tfrac{1}{2}] \text{ and } f(x) = 0, \text{ if } x \in (\tfrac{1}{2}, 1],$$

$$g(x) = \tfrac{1}{2}, \text{ if } x \in [0, \tfrac{1}{2}] \text{ and } g(x) = \tfrac{3}{4}, \text{ if } x \in (\tfrac{1}{2}, 1].$$

Then, for the sequence $\{x_n\} = \{\frac{1}{2} - \frac{1}{n}\}$, $n \geq 2$, we have $lim_{n\to\infty} f(\frac{1}{2} - \frac{1}{n}) = lim_{n\to\infty} \frac{1}{2} + \frac{1}{n} = \frac{1}{2} = lim_{n\to\infty} g(\frac{1}{2} - \frac{1}{n})$. Thus, the pair (f, g) satisfies property $(E.A)$. Further, f and g are weakly compatible since $x = \frac{1}{2}$ is their unique coincidence point and $fg(1/2) = f(1/2) = g(1/2) = gf(1/2)$. We further observe that $lim_{n\to\infty} d(fg(\frac{1}{2} - \frac{1}{n}), gf(\frac{1}{2} - \frac{1}{n})) = lim_{n\to\infty} d(f(\frac{1}{2}), g(\frac{1}{2} + \frac{1}{n})) = d(\frac{1}{2}, \frac{3}{4}) \neq 0$,showing that the pair (f, g) is noncompatible.

Example 2.2. Let $X = \mathbb{R}_+$ and d be the usual metric on X. Define $f, g : X \to X$ by:

$fx = 0$, if $0 < x \leq 1$ and $fx = 1$, if $x > 1$ or $x = 0$; and

$gx = [x]$, the greatest integer that is less than or equal to x, $\forall x \in X$.

Consider a sequence $\{x_n\} = \{1 + \frac{1}{n}\}_{n\geq 2}$ in (1, 2), then we have $lim_{n\to\infty} fx_n = 1 = lim_{n\to\infty} gx_n$. Similarly for the sequence $\{y_n\} = \{1 - \frac{1}{n}\}_{n\geq 2}$ in (0, 1), we have $lim_{n\to\infty} fy_n = 0 = lim_{n\to\infty} gy_n$. Thus the pair (f, g) satisfies(E.A). However, f and g are not weakly compatible; as each $u_1 \in (0, 1)$ and $u_2 \in (1, 2)$ are coincidence points of f and g, where they do not commute. Moreover, they commute at $x = 0, 1, 2, ...$ but none of these points are coincidence points of f and g. Further, (f, g) is noncompatible. Hence, (E.A) $\not\Longrightarrow$ weak compatibility.

3. Implicit relation

Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real and non-negative real numbers, respectively. We now state an implicit relation [6] as follows:

Let $\mathcal{F}$ be the set of all continuous functions F with $F(t_1, ..., t_6) : {\mathbb{R}_+}^6 \to \mathbb{R}$ satisfying the following conditions:

(F_1) : $F(u,\ 0,\ u,\ 0,\ 0,\ u) \leq 0 \Longrightarrow u = 0,$ (3.1)

(F_2) : $F(u,\ 0,\ 0,\ u,\ u,\ 0) \leq 0 \Longrightarrow u = 0.$ (3.2)

The function $F(t_1, ..., t_6) : {\mathbb{R}_+}^6 \to \mathbb{R}$ is said to satisfy condition (F_u) if:

(F_u) : $F(u,\ u,\ 0,\ 0,\ u,\ u) \geq 0, \quad \forall u > 0.$ (3.3)

The following are examples of the implicit relation defined above. Another examples can be found in [5-6].

Example 3.1. Let $F(t_1, ..., t_6) = pt_1 - qt_2 - r(t_3 + t_4) - s(t_5 + t_6)$, where $p, q, r, s \geq 0, \quad 0 \leq r + s < p$ and $0 \leq q + 2s \leq p$, then:

(F_1) : $F(u, 0, u, 0, 0, u) = u(p - r - s) \leq 0$ implies $u = 0$;

(F_2) : $F(u, 0, 0, u, u, 0) = u(p - r - s) \leq 0$ implies $u = 0$ and

(F_u) : $F(u, u, 0, 0, u, u) = u(p - q - 2s) \geq 0, \quad \forall u > 0.$

Example 3.2. Let $F(t_1, ..., t_6) = pt_1 - max\{qt_2, (t_3 + t_4)/2, s(t_5 + t_6)/2\}$, where $0 \leq s < q < 1/2 < p$, then:

(F_1) : $F(u, 0, u, 0, 0, u) = pu - max\{u/2, su/2\} = u(p - 1/2) \leq 0 \Rightarrow u = 0;$

(F_2) : $F(u, 0, 0, u, u, 0) = pu - max\{u/2, su/2\} = u(p - 1/2) \leq 0 \Rightarrow u = 0;$

(F_u) : $F(u, u, 0, 0, u, u) = pu - max\{qu, 0, su\} = u(p - q) \geq 0, \quad \forall u > 0.$

Example 3.3. Let $F(t_1, ..., t_6) = t_1 - max\{qt_2, r(t_3 + t_4)/2, (t_5 + t_6)/2\}$, where $0 \leq q < 1 \leq r < 2$, then:

(F_1) : $F(u, 0, u, 0, 0, u) = u - max\{0, ru/2, u/2\} = u(1 - r/2) \leq 0 \Rightarrow u = 0;$

(F_2) : $F(u, 0, 0, u, u, 0) = u - max\{0, ru/2, u/2\} = u(1 - r/2) \leq 0 \Rightarrow u = 0;$

(F_u) : $F(u, u, 0, 0, u, u) = u - max\{qu, 0, u\} = u - u = 0, \quad \forall u > 0.$

Example 3.4. Let $F(t_1, ..., t_6) = t_1 - h\,max\{t_2, t_3, t_4, t_5, t_6\}$, where $0 \leq h < 1$, then:

(F_1) : $F(u, 0, u, 0, 0, u) = u - h\,max\{0, u, 0, 0, u\} = u(1 - h) \leq 0 \Rightarrow u = 0;$

(F_2) : $F(u, 0, 0, u, u, 0) = u - h\,max\{0, 0, u, u, 0\} = u(1 - h) \leq 0 \Rightarrow u = 0;$

(F_u) : $F(u, u, 0, 0, u, u) = u - h\,max\{u, 0, 0, u, u\} = u(1 - h) \geq 0, \quad \forall u > 0.$

Example 3.5. Let $F(t_1, ..., t_6) = t_1^2 - at_2^2 - t_3t_4 - bt_5^2 - ct_6^2$ where $a, b, c \geq 0$

and $0 < a + b + c < 1$ then:
$(F_1): \quad F(u, 0, u, 0, 0, u) = u^2(1 - c) \leq 0 \Rightarrow u = 0;$
$(F_2): \quad F(u, 0, 0, u, u, 0) = u^2(1 - b) \leq 0 \Rightarrow u = 0$ and

$(F_u): \quad F(u, u, 0, 0, u, u) = u^2(1 - a - b - c) \geq 0, \quad \forall u > 0.$
Example 3.6. Let $F(t_1, ..., t_6) = t_1^3 - k(t_2^3 + t_3^3 + t_4^3 + t_5^3 + t_6^3)$, where $0 \leq k \leq 1/3$,then:
$(F_1): \quad F(u, 0, u, 0, 0, u) = u^3(1 - 2k) \leq 0 \Rightarrow u = 0;$

$(F_2): \quad F(u, 0, 0, u, u, 0) = u^3(1 - 2k) \leq 0 \Rightarrow u = 0$ and
$(F_u): \quad F(u, u, 0, 0, u, u) = u^3(1 - 3k) \geq 0, \quad \forall u > 0.$
The following lemma is useful to prove the existence of common fixed point.

Lemma 3.7 [6]. *Let (X, d) be a metric space and A, B, S and T be four self-mappings on X satisfying:*

$$F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(By, Sx), d(Ax, Ty)) < 0,$$

for all $x, y \in X$, where F satisfies property (F_u). Then A, B, S and T have at most one common fixed point.

The following theorem was proved by Popa [6] for a Meir-Keeler type contractive condition using the implicit relation:
Theorem A. ([6]) *Let A, B, S and T be self-mappings of a metric space (X, d) such that*

(a) $A(X) \subseteq T(X), \quad B(X) \subseteq S(X),$

(b) given $\epsilon > 0$, there exists $\delta > 0$ such that
$\epsilon \leq max \{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{1}{2}[d(By, Sx) + d(Ax, Ty))]\} < \epsilon + \delta \Longrightarrow d(Ax, By) < \epsilon,$
(c) there exists $F \in \mathcal{F}$ such that the inequality:
$F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(By, Sx), d(Ax, Ty)) < 0,$

holds for all $x, y \in X$.

If one of $A(X)$, $B(X)$, $S(X)$ or $T(X)$ is a complete subspace of X then,
(d) A and S have a coincidence point.
(e) B and T have a coincidence point.

Moreover, if the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

The following theorem was proved by Aamri and Moutawakil [1] under property (E.A)using a contractive condition:

Theorem B. ([1]) *Let A, B, S and T be self-mappings of a metric space (X, d) such that*
(a) $d(Ax, By) \leq \phi(max\,\{d(Sx, Ty), d(By, Sx), d(By, Ty)\}, \quad \forall (x, y) \in X^2$,
where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a non-decreasing function on* $\mathbb{R}_+$ *such that* $0 < \phi(t) < t$*, for each* $t \in (0, \infty)$.
(b) (A, S) *and* (B, T) *are weakly compatible,*
(c) (A, S) *or* (B, T) *satisfy property (E.A),*
(d) $A(X) \subseteq T(X), \quad B(X) \subseteq S(X)$.

If the range of one of the mappings is a complete subspace of X*, then* A*,* B*,* S*and* T *have a unique common fixed point.*

In this paper, we intend to unify Theorem A and Theorem B by imposing property (E.A).Theorem A uses the Meir-Keeler type contractive condition which is to be removed by an independent notion viz. property (E.A). Similarly, Theorem B uses a ϕ-contractive condition which is to be removed by its generalized condition viz. implicit relation. Thus we will use property (E.A) of [1] and implicit relation of [6] to unify under property(E.A) and implicit relation.

The following two examples show that Meir-Keeler type contractive condition and property (E.A) are independent to each other.

Example 3.8. Let A, B, S and T be four self-mappings of the metric space $([0, 1], d)$ with the usual metric d defined by
$Ax = Bx = 0$, if $x = 0$ or $x = 1$, $\;Ax = Bx = \frac{1}{n+1}$,if $\frac{1}{n+1} \leq x < \frac{1}{n}$, $n \in \mathbb{N}$;
and
$Sx = Tx = x, \quad \forall x \in X$.
The Meir-Keeler type contractive condition is defined by:
given $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$,

$$\epsilon \leq M(x, y) < \epsilon + \delta \Longrightarrow d(Ax, By) < \epsilon. \qquad (MKC)$$

where

$$M(x, y) = max\,\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{1}{2}[d(By, Sx) + d(Ax, Ty))]\}.$$

Let us discuss property (E.A) and MKC condition for various cases.
Here, (B, T) satisfies property (E.A). Indeed, taking $\{\frac{1}{n+1}\} \subseteq [0, 1]$, we get$lim_{n\to\infty} B(\frac{1}{n+1}) = lim_{n\to\infty}\frac{1}{n+1} = 0 = lim_{n\to\infty} T(\frac{1}{n+1}) = lim_{n\to\infty}\frac{1}{n+1}$.
Similarly, (A,S) satisfies property (E.A).
Next, we check that property MKC is not valid.

(i) If $x = 0$ and $y = \frac{1}{n}$, $n \geq 2$, then $x \neq y$ and$d(Ax, By) = d(0, \frac{1}{n}) = \frac{1}{n}$. Further, $\epsilon \leq M(x, y) < \epsilon + \delta$ yields

$$\epsilon \leq \max \left\{ d\left(0, \frac{1}{n}\right), d(0,0), d\left(\frac{1}{n}, \frac{1}{n}\right), \frac{1}{2}\left(d\left(\frac{1}{n}, 0\right) + d\left(\frac{1}{n}, 0\right)\right)\right\}$$
$$= \frac{1}{n} < \epsilon + \delta,$$

showing that $\epsilon \leq d(Ax, By) = \frac{1}{n}$, contradicting MKC condition.
(ii) Similarly, if $y = 0$ and $x = \frac{1}{n+1} \neq 0$, a similar argument verifies that the MKC condition(b) is not satisfied: if

$$\epsilon \leq \max \left\{ d\left(\frac{1}{n+1}, 0\right), d\left(\frac{1}{n+1}, \frac{1}{n+1}\right), d(0,0), \right.$$
$$\left. \frac{1}{2}\left(d\left(0, \frac{1}{n+1}\right) + d\left(\frac{1}{n+1}, 0\right)\right)\right\}$$
$$= \frac{1}{n+1} < \epsilon + \delta,$$

showing that $\epsilon \leq d(Ax, By) = \frac{1}{n+1}$, contradicting MKC condition.
(iii) On the other hand, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that for $x = x_n = y_n = y \in [\frac{1}{n+1}, \frac{1}{n})$, we have that $\epsilon \leq M(x, y) < \epsilon + \delta$ yields

$$\epsilon \leq max \left\{ 0, \left|\frac{1}{n+1} - x_n\right|, \left|\frac{1}{n+1} - x_n\right|, \frac{1}{2}\left[\left|\frac{1}{n+1} - x_n\right| + \left|\frac{1}{n+1} - x_n\right|\right]\right\}$$
$$= \left|\frac{1}{n+1} - x_n\right| < \epsilon + \delta,$$

implies $d(Ax_n, By_n) = d\left(\frac{1}{n+1}, \frac{1}{n+1}\right) = 0 < \epsilon$. For these points,MKC condition is satisfied.
This example illustrates the fact that property (E.A)$\not\Longrightarrow$ M.K.C.
Besides, the pair $(A, S)$is weakly compatible, since $Ax = Sx$ occurs if $x = 0$ or $x = \frac{1}{n+1}$, $n \in \mathbb{N}$, in such cases, $AS0 = A0 = 0 = S0 = SA0$and$AS(\frac{1}{n+1}) = A(\frac{1}{n+1}) = \frac{1}{n+1} = S(\frac{1}{n+1}) = SA(\frac{1}{n+1})$, $n \in \mathbb{N}$. Similarly, $(B, T)$is weakly compatible.

Example 3.9. Consider the metric space $([0, 1], d)$ with the usual metric d and let A, B, S and T be four self-mappings of the metric space Xdefined by $Ax = Bx = 0$, if $x \in X$; and $Sx = Tx = 1$, $\forall x \in X$.
For every sequence $\{x_n\}$ in X, we get $lim_{n\to\infty} Ax_n = 0 \neq 1 = lim_{n\to\infty} Sx_n$,

and analogously for the pair (B,T). Hence, property (E.A)is not satisfied for the pairs (A,S), (B,T).
On the other hand,given $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$,

$$\epsilon \leq M(x,y) < \epsilon + \delta \Longrightarrow d(Ax, By) < \epsilon.$$

Indeed,

$$M(x,y) = max\,\{d(1,1), d(0,1), d(0,1), \frac{1}{2}[d(0,1) + d(0,1))]\} = 1.$$

Hence, for $\epsilon > 1$, there are no x, y satisfying $\epsilon \leq M(x,y) = 1 < \epsilon + \delta$. On the other hand, if$\epsilon \leq 1$, and we take $\delta > 1 - \epsilon \geq 0$, for x, y with $\epsilon \leq M(x,y) = 1 < \epsilon + \delta$, it is obvious that $d(Ax, By) = 0 < \epsilon$.
Therefore the Meir-Keeler type contractive condition (MKC) does not imply property (E.A). Symbolically, MKC $\not\Longrightarrow$ property (E.A).
Hence Meir-Keeler type contractive condition and property (E.A) are independent to each other.
Next, we show an example where both conditions (E.A) and (MKC) are satisfied.
Example 3.10. Let $X = [0,1]$, and for all $x, y \in X$, $d(x,y) = max\,\{x, y\}$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$. One can verify that $(X,d)$is a complete metric space. Further, define mappings $A, B, S, T : X \to X$ by:
$Ax = Bx = 0$, if $x = 0$, and $Ax = Bx = \frac{1}{n+1}$, if $\frac{1}{n+1} < x \leq \frac{1}{n}$, $n \in \mathbb{N}$, and
$Sx = Tx = 0$, if $x = 0$, and $Sx = Tx = \frac{1}{n}$, if $\frac{1}{n+1} < x \leq \frac{1}{n}$, $n \in \mathbb{N}$.
If, for a fixed $n \in \mathbb{N}$, we consider the sequence$\{y_k\} = \{\frac{1}{n+1} + \frac{1}{3^k}\left(\frac{1}{n} - \frac{1}{n+1}\right)\}$ in the interval$(\frac{1}{n+1}, \frac{1}{n}]$, then$lim_{k\to\infty} d(By_k, Ty_k) = \frac{1}{n} \neq 0$,$lim_{k\to\infty} By_k = \frac{1}{n+1}$ and$lim_{k\to\infty} Ty_k = \frac{1}{n}$, since $d(By_k, \frac{1}{n+1}) = 0$ and $d(Ty_k, \frac{1}{n}) = 0$, for every k. Thus property (E.A) can not be extracted using such a sequence. However, property (E.A) holds, and we can check it just by taking the sequence in X given by $\{y_n\} = \{\frac{1}{n}\}$, which satisfies $lim_{n\to\infty} d(By_n, 0) = lim_{n\to\infty} d(\frac{1}{n+1}, 0) = lim_{n\to\infty} \frac{1}{n+1} = 0$ and

$$lim_{n\to\infty} d(Ty_n, 0) = lim_{n\to\infty} d(\frac{1}{n}, 0) = lim_{n\to\infty} \frac{1}{n} = 0.$$

Hence $lim_{n\to\infty} By_n = lim_{n\to\infty} Ty_n = 0 \in X$ and (B,T) satisfies property (E.A). Similar considerations can be made for the pair (A,S).
Next, we check that the Meir-Keeler type contractive condition (MKC) is valid. Let $\epsilon > 0$. Let

$$M(x,y) = max\,\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \frac{1}{2}[d(By,Sx) + d(Ax,Ty)]\},$$

and observe the following:
(i) If $x = y = 0$, then $M(x, y) = 0$ and $d(Ax, By) = 0$. Note that, in this case, it is not possible that $\epsilon \leq M(x, y)$.
(ii) If $x = 0$ and $y \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$, then

$$\begin{aligned} &M(x,y)\\ &\quad = max\left\{d\left(0,\frac{1}{n}\right), d(0,0), d\left(\frac{1}{n+1},\frac{1}{n}\right), \frac{1}{2}\left(d\left(\frac{1}{n+1},0\right)+d\left(0,\frac{1}{n}\right)\right)\right\}\\ &\quad = max\left\{\frac{1}{n}, 0, \frac{1}{n}, \frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n}\right)\right\} = \frac{1}{n}. \end{aligned}$$

Besides, $d(Ax, By) = d\left(0, \frac{1}{n+1}\right) = \frac{1}{n+1}$.
(iii) If $y = 0$ and $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$, then

$$\begin{aligned} &M(x,y)\\ &\quad = max\left\{d\left(\frac{1}{n},0\right), d\left(\frac{1}{n+1},\frac{1}{n}\right), d(0,0), \frac{1}{2}\left(d\left(0,\frac{1}{n}\right)+d\left(\frac{1}{n+1},0\right)\right)\right\}\\ &\quad = max\left\{\frac{1}{n}, \frac{1}{n}, 0, \frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)\right\} = \frac{1}{n}, \end{aligned}$$

and $d(Ax, By) = d\left(\frac{1}{n+1}, 0\right) = \frac{1}{n+1}$.
(iv) If $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$ and $y \in \left(\frac{1}{m+1}, \frac{1}{m}\right]$, with $n < m$, then

$$\begin{aligned} &M(x,y)\\ &\quad = max\left\{d\left(\frac{1}{n},\frac{1}{m}\right), d\left(\frac{1}{n+1},\frac{1}{n}\right), d\left(\frac{1}{m+1},\frac{1}{m}\right),\right.\\ &\quad \left.\frac{1}{2}\left(d\left(\frac{1}{m+1},\frac{1}{n}\right)+d\left(\frac{1}{n+1},\frac{1}{m}\right)\right)\right\}\\ &\quad = max\left\{\frac{1}{n}, \frac{1}{n}, \frac{1}{m}, \frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)\right\} = \frac{1}{n}, \end{aligned}$$

and $d(Ax, By) = d\left(\frac{1}{n+1}, \frac{1}{m+1}\right) = \frac{1}{n+1}$.
On the other hand, if $n > m$,we get

$$M(x,y) = max\left\{\frac{1}{m}, \frac{1}{n}, \frac{1}{m}, \frac{1}{2}\left(\frac{1}{m+1}+\frac{1}{m}\right)\right\} = \frac{1}{m},$$

and $d(Ax, By) = d\left(\frac{1}{n+1}, \frac{1}{m+1}\right) = \frac{1}{m+1}$.

(v) If $x,\, y \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$, then

$$\begin{aligned} &M(x,y)\\ &\quad = max\left\{d\left(\frac{1}{n},\frac{1}{n}\right), d\left(\frac{1}{n+1},\frac{1}{n}\right), d\left(\frac{1}{n+1},\frac{1}{n}\right),\right.\\ &\quad \left.\frac{1}{2}\left(d\left(\frac{1}{n+1},\frac{1}{n}\right)+d\left(\frac{1}{n+1},\frac{1}{n}\right)\right)\right\}\\ &\quad = max\left\{\frac{1}{n},\frac{1}{n},\frac{1}{n},\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n}\right)\right\} = \frac{1}{n}, \end{aligned}$$

and $d(Ax, By) = d\left(\frac{1}{n+1}, \frac{1}{n+1}\right) = \frac{1}{n+1}$.

Therefore, it suffices to find $\delta > 0$ such that $\epsilon \leq M(x,y) = \frac{1}{n} < \epsilon+\delta$ implies $d(Ax, By) = \frac{1}{n+1} < \epsilon$.If $\epsilon > 1$, there are no x, y such that $\epsilon \leq M(x,y) = \frac{1}{n}$. If $\epsilon = 1$, then $\epsilon \leq M(x,y) = \frac{1}{n}$ implies that $n = 1$ and hence it is obvious that $d(Ax, By) = \frac{1}{n+1} = \frac{1}{2} < \epsilon$.If we fix $0 < \epsilon < 1$, then there exists a finite number of integer numbers satisfying that $\epsilon \leq M(x,y) = \frac{1}{n}$. For these numbers, $\frac{1}{n} < \epsilon+\delta$ is equivalent to $\frac{1}{n} - \delta < \epsilon$, so that, if we get that $d(Ax, By) = \frac{1}{n+1} \leq \frac{1}{n} - \delta$, for n with $\epsilon \leq \frac{1}{n}$,we deduce MKC condition.Then, it suffices to take $0 < \delta \leq \frac{1}{n} - \frac{1}{n+1}$, for n such that $\frac{1}{\epsilon} \geq n$. We can choose, for instance, $0 < \delta = \frac{1}{K} - \frac{1}{K+1} = \frac{1}{K(K+1)}$, where $K = \left[\frac{1}{\epsilon}\right]$(the integer part of $\frac{1}{\epsilon}$).
This proves that MKC is satisfied.

In our main Theorem we will apply property (E.A) and implicit relation. Instead of condition (b) viz. Meir-Keeler type contractive condition of Theorem A, we will impose property (E.A). Similarly, we will use more general contractive condition than (a) as used in Theorem B. So we are unifying the Theorem A and Theorem B and generalizing the Theorem B for an implicit relation and property (E.A). In this paper, we prove a common fixed point theorem for a quadruple of mappings by using an implicit relation and property (E.A) under weak compatibility. By an example we illustrate and verify our main Theorem. Here is our Main Result:

4. Main Results

In this section we state and prove our main result.

Theorem 4.1. *Let A, B, S and T be four self-maps of a metric space (X,d) satisfying:*
(i) $A(X) \subseteq T(X), \quad B(X) \subseteq S(X)$,
(ii) there exists a continuous function $F : {\mathbb{R}_+}^6 \to \mathbb{R}$ *in* $\mathcal{F}$ *such that*

$$F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(By, Sx), d(Ax, Ty)) < 0,$$

for all $x, y \in X$, where F satisfies all the conditions of implicit relation,
(iii) (A,S) and (B,T) are weakly compatible,
(iv) (A,S) or (B,T) satisfy property (E.A).
Assume that one of the following conditions hold:
(v) $\{By_n\}$ is a bounded sequence for every $\{y_n\} \subseteq X$ such that$\{Ty_n\}$ is convergent (in case (A,S) satisfies property (E.A)), and $\{Ay_n\}$ is a bounded sequence for every $\{y_n\} \subseteq X$ such that$\{Sy_n\}$ is convergent (in case (B,T) satisfies property (E.A)).
(vi) If $\{z_n\}$, $\{r_n\}$ and $\{w_n\}$ are nonnegative sequences such that $\{z_n\} \to \infty$, $\{w_n\} \to \infty$, as $n \to \infty$ and

$$F(z_n, r_n, r_n, z_n, w_n, 0) \leq 0,\ n \in \mathbb{N}, \text{(in case } (A,S) \text{ satisfies (E.A))},$$

$$F(z_n, r_n, z_n, r_n, 0, w_n) \leq 0,\ n \in \mathbb{N}, \text{(in case } (B,T) \text{ satisfies (E.A))},$$

then $\{r_n\} \to \infty$, as $n \to \infty$.

If the range of one of the mappings is a complete subspace of X, then A, B, S and T have a unique common fixed point.

Proof. Suppose that (B,T) satisfies property (E.A) then, by definition, there exists a sequence $\{x_n\}$in X such that $lim_{n\to\infty} Bx_n = lim_{n\to\infty} Tx_n = t$, for some $t \in X$. Since $B(X) \subseteq S(X)$, there exists in X a sequence $\{y_n\}$ such that $Bx_n = Sy_n$. Hence $lim_{n\to\infty} Sy_n = t$. Let us show that $lim_{n\to\infty} Ay_n = t$. Indeed, in view of implicit relation (ii), we have

$$F(d(Ay_n, Bx_n),\ d(Sy_n, Tx_n),\ d(Ay_n, Sy_n),\ d(Bx_n, Tx_n),\ d(Bx_n, Sy_n),\ d(Ay_n, Tx_n)) < 0,$$

i.e.,

$$F(d(Ay_n, Bx_n),\ d(Bx_n, Tx_n),\ d(Ay_n, Bx_n),\ d(Bx_n, Tx_n),\ d(Bx_n, Bx_n),\ d(Ay_n, Tx_n)) \leq 0,$$

or,

$$F(d(Ay_n, Bx_n),\ d(Bx_n, Tx_n),\ d(Ay_n, Bx_n),\ d(Bx_n, Tx_n),\ 0,\ d(Ay_n, Tx_n)) \leq 0.$$

Using condition *(v)*, $\{Ay_n\}$ is a bounded sequence. If condition *(vi)* holds, and we suppose that $\{Ay_n\}$ is not bounded, then there exists a subsequence $\{n_k\}$ of nonnegative integer numbers such that $d(Ay_{n_k}, Bx_{n_k}) \to +\infty$, as $k \to \infty$. Similarly, $d(Ay_{n_k}, Tx_{n_k}) \to +\infty$, as $k \to \infty$, and, by the implicit relation and *(vi)*, $d(Bx_{n_k}, Tx_{n_k}) \to +\infty$, as $k \to \infty$, which is a contradiction. Hence, in both cases, $\{Ay_n\}$ is a bounded sequence and $\{d(Ay_n, Bx_n)\}$ is bounded,hence$limsup_{n\to\infty} d(Ay_n, Bx_n)$ is a finite number.Consider $p_n = d(Ay_n, Bx_n)$, $c_n = d(Bx_n, Tx_n)$ and $q_n = d(Ay_n, Tx_n)$, $n \in \mathbb{N}$. Note that

$$|p_n - q_n| = |d(Ay_n, Bx_n) - d(Ay_n, Tx_n)| \leq d(Bx_n, Tx_n) \to 0,\ n \to \infty,$$

in consequence,$|p_n - q_n| \to 0$, $n \to \infty$.Since $\{p_n\}$ is bounded (and $\{q_n\}$ is bounded), the numbers $limsup_{n\to\infty} p_n$ and $limsup_{n\to\infty} q_n$ are finite. Indeed, in this case, $limsup_{n\to\infty} p_n = limsup_{n\to\infty} q_n$. To check that $limsup_{n\to\infty} p_n$ is the upper limit of $\{q_n\}$, it suffices to check that, if $\{p_{n_k}\} \to p$ then $\{q_{n_k}\} = \{q_{n_k} - p_{n_k}\} + \{p_{n_k}\} \to p$, and conversely.Besides, $\{c_n\} \to 0$, as $n \to \infty$. Using the continuity of F and denoting $limsup_{n\to\infty} d(Ay_n, Bx_n) = limsup_{n\to\infty} p_n = l$, we check that

$$F(l,\ 0,\ l,\ 0,\ 0,\ l) \leq 0.$$

Indeed, using that $limsup_{n\to\infty} p_n = l$, we obtain a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $\{p_{n_k}\} \to l$, as $k \to \infty$, hence$\{q_{n_k}\} = \{q_{n_k} - p_{n_k}\} + \{p_{n_k}\} \to l$, and $\{c_{n_k}\} \to 0$, as $k \to \infty$. Since

$$F(p_{n_k}, c_{n_k}, p_{n_k}, c_{n_k}, 0, q_{n_k}) \leq 0,\ \forall k \in \mathbb{N},$$

using continuity of F, we get $F(l,\ 0,\ l,\ 0,\ 0,\ l) \leq 0$.
Using (F_1), we get $l = limsup_{n\to\infty} d(Ay_n, Bx_n) = 0$, and $lim_{n\to\infty} d(Ay_n, Bx_n) = 0$, whence $lim_{n\to\infty} Ay_n = t$. Hence in all $Ay_n \to t,\ Sy_n \to t,\ Bx_n \to t,$ and $Tx_n \to t,$ as $n \to \infty$, for some $t \in X$.

Next, suppose that $S(X)$ is a complete subspace of X, then $t = Su$, for some $u \in X$.In order to show that $Au = t$, putting u for x and x_n for y in the implicit relation (ii), we have

$$F(d(Au, Bx_n),\ d(Su, Tx_n),\ d(Au, Su),\ d(Bx_n, Tx_n),\ d(Bx_n, Su),\ d(Au, Tx_n)) < 0,$$

letting $n \to \infty$ it yields

$$F(d(Au, t),\ 0,\ d(Au, t),\ 0,\ 0,\ d(Au, t)) \leq 0,$$

which, on using (F_1), yields $d(Au, t) = 0$, so that $Au = t = Su$ and u is a coincidence point of A and$S$.

Now, since $A(X) \subseteq T(X)$, there exists $v \in X$ such that $Au = Tv$. We claim that $Bv = t$. Using implicit relation (ii) we have,

$$F(d(Au, Bv),\ d(Su, Tv),\ d(Au, Su),\ d(Bv, Tv),\ d(Bv, Su),\ d(Au, Tv)) < 0,$$

i.e., $$F(d(t, Bv),\ 0,\ 0,\ d(Bv, t),\ d(Bv, t),\ 0) < 0. \qquad (A)$$

Similarly, by putting y_n for x and v for y in the implicit relation, we have

$$F(d(Ay_n, Bv),\ d(Sy_n, Tv),\ d(Ay_n, Sy_n),\ d(Bv, Tv),\ d(Bv, Sy_n),\ d(Ay_n, Tv)) < 0,$$

which, on letting $n \to \infty$, gives

$$F(d(t, Bv),\ 0,\ 0,\ d(Bv, t),\ d(Bv, t),\ 0) \leq 0 \qquad (B)$$

from (A) and (B), on using (F_2), we have $d(t, Bv) = 0$, i.e., $Bv = t = Tv$ and v is a coincidence point of$B$ and T. Thus, $Au = Su = Bv = Tv = t$.

The weak compatibility of A and S implies that $ASu = SAu$ so that $AAu = ASu = SAu = SSu$. Let us show that Au is a common fixed point of A and S. If $AAu \neq Au$, then implicit relation (ii) yields

$$F(d(AAu, Bv),\ d(SAu, Tv),\ d(AAu, SAu),\ d(Bv, Tv),\ d(Bv, SAu),\ d(AAu, Tv)) < 0,$$

i.e., $F(d(AAu, Au),\ d(AAu, Au),\ 0,\ 0,\ d(Au, AAu),\ d(AAu, Au)) < 0$, a contradiction of (F_u). Thus $AAu = Au = SAu$, and Au is a common fixed point of A and S. Similarly we can prove that Bv is a common fixed point of B and T. Since $Au = Bv$, we conclude that Au is a common fixed point of A, B, S and T.

The proof is similar when $T(X)$ is assumed to be a complete subspace of X. The cases in which $A(X)$ or $B(X)$ is a complete subspace of X are similar to the cases in which $T(X)$ or $S(X)$ respectively is complete, since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.

The uniqueness follows from Lemma 3.7. This proves the existence of a unique common fixed point of A, B, S and$T$. This completes the proof.

Remark 4.2. Note that condition *(vi)* in Theorem 4.1 can be expressed similarly in the following way: if$\{z_n\}$,$\{r_n\}$ and $\{w_n\}$ are nonnegative sequences such that $limsup_{n\to\infty} z_n = limsup_{n\to\infty} w_n = \infty$, and

$$F(z_n, r_n, r_n, z_n, w_n, 0) \leq 0,\ n \in \mathbb{N}, \text{(in case } (A, S) \text{ satisfies (E.A))},$$

$$F(z_n, r_n, z_n, r_n, 0, w_n) \leq 0,\ n \in \mathbb{N}, \text{(in case } (B, T) \text{ satisfies (E.A))},$$

then $limsup_{n\to\infty} r_n = \infty$.

Remark 4.3. Continuity of F is a very restrictive condition in Theorem 4.1. We can relax continuity of$F$, extending this result to the more general case where$F$ is continuous at certain points of the boundary of $(\mathbb{R}_+)^6$. Consider the following conditions:

(C) If $\{p_n\}$, $\{c_n\}$ and $\{q_n\}$ are nonnegative sequences such that $\{p_n\}$ is bounded, $\{|p_n - q_n|\} \to 0$, as $n \to \infty$, $\{c_n\} \to 0$, as $n \to \infty$, and

$$F(p_n,\ \ c_n,\ \ p_n,\ \ c_n,\ \ 0,\ \ q_n) \leq 0,\ n \in \mathbb{N},$$

then$F\left(\limsup_{n\to\infty} p_n,\ \ 0,\ \ \limsup_{n\to\infty} p_n,\ \ 0,\ \ 0,\ \ \limsup_{n\to\infty} q_n\right) \leq 0.$

$(\widehat{\text{C}})$ If $\{p_n\}$, $\{c_n\}$ and $\{q_n\}$ are nonnegative sequences such that $\{p_n\}$ is bounded, $\{|p_n - q_n|\} \to 0$, as $n \to \infty$, $\{c_n\} \to 0$, as $n \to \infty$, and

$$F(p_n,\ \ c_n,\ \ c_n,\ \ p_n,\ \ q_n,\ \ 0) \leq 0,\ n \in \mathbb{N},$$

then$F\left(\limsup_{n\to\infty} p_n,\ \ 0,\ \ 0,\ \ \limsup_{n\to\infty} p_n,\ \ \limsup_{n\to\infty} q_n,\ \ 0\right) \leq 0.$

Alternatively, we can use conditions (CS) and $(\widehat{\text{CS}})$ expressed in terms of

sequences:
(CS) If $\{p_n\}$, $\{c_n\}$ and $\{q_n\}$ are nonnegative sequences such that $\{p_n\} \to l$, $\{q_n\} \to l$, $\{c_n\} \to 0$, as $n \to \infty$, and

$$F(p_n,\ \ c_n,\ \ p_n,\ \ c_n,\ \ 0,\ \ q_n) \leq 0,\ n \in \mathbb{N},$$

then$F\,(l,\ \ 0,\ \ l,\ \ 0,\ \ 0,\ \ l) \leq 0$.
($\widehat{\text{CS}}$) If $\{p_n\}$, $\{c_n\}$ and $\{q_n\}$ are nonnegative sequences such that $\{p_n\} \to l$, $\{q_n\} \to l$, $\{c_n\} \to 0$, as $n \to \infty$, and

$$F(p_n,\ \ c_n,\ \ c_n,\ \ p_n,\ \ q_n,\ \ 0) \leq 0,\ n \in \mathbb{N},$$

then$F\,(l,\ \ 0,\ \ 0,\ \ l,\ \ l,\ \ 0) \leq 0$.
Note that (CS) is trivially valid if $\tilde{F}(t_1, t_2, t_3) = F(t_1, t_2, t_1, t_2, 0, t_3)$ is continuous at $(z, 0, z)$, $z \in \mathbb{R}_+$, or if F restricted to the set $(\mathbb{R}_+)^4 \times \{0\} \times \mathbb{R}_+$ is continuous at$(z, 0, z, 0, 0, z)$, $z \in \mathbb{R}_+$. Besides, ($\widehat{\text{CS}}$) is trivially valid if$\tilde{F}(t_1, t_2, t_3) = F(t_1, t_2, t_2, t_1, t_3, 0)$ is continuous at $(z, 0, z)$, $z \in \mathbb{R}_+$, or if F restricted to the set $(\mathbb{R}_+)^5 \times \{0\}$ is continuous at$(z, 0, 0, z, z, 0)$, $z \in \mathbb{R}_+$.
Consider also the following conditions:(H1) If $\{a_n\}$, $\{b_n\}$, $\{d_n\}$, $\{e_n\}$, and $\{f_n\}$ are nonnegative sequences such that $\{a_n\} \to l$, $\{f_n\} \to l$, $\{b_n\} \to 0$, $\{d_n\} \to 0$, $\{e_n\} \to 0$, as $n \to \infty$, and

$$F(a_n,\ \ b_n,\ \ l,\ \ d_n,\ \ e_n,\ \ f_n) \leq 0,\ n \in \mathbb{N},$$

then$F\,(l,\ \ 0,\ \ l,\ \ 0,\ \ 0,\ \ l) \leq 0$.
(H2) If $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{e_n\}$, and $\{f_n\}$ are nonnegative sequences such that $\{a_n\} \to l$, $\{e_n\} \to l$, $\{b_n\} \to 0$, $\{c_n\} \to 0$, $\{f_n\} \to 0$, as $n \to \infty$, and

$$F(a_n,\ \ b_n,\ \ c_n,\ \ l,\ \ e_n,\ \ f_n) \leq 0,\ n \in \mathbb{N},$$

then$F\,(l,\ \ 0,\ \ 0,\ \ l,\ \ l,\ \ 0) \leq 0$.
Note that (H1) is valid if F is continuous at $(z, 0, z, 0, 0, z)$, $z \in \mathbb{R}_+$, or if each restriction of F to the subset $\{t_3 = z\}$ is continuous at $(z, 0, z, 0, 0, z)$, $z \in \mathbb{R}_+$, that is,$F_z(t_1, t_2, t_4, t_5, t_6) = F(t_1, t_2, z, t_4, t_5, t_6)$ is continuous at $(z, 0, 0, 0, z)$, $z \in \mathbb{R}_+$.Similarly, (H2) is valid if F is continuous at $(z, 0, 0, z, z, 0)$, $z \in \mathbb{R}_+$, or if each restriction of F to the subset $\{t_4 = z\}$ is continuous at $(z, 0, 0, z, z, 0)$, $z \in \mathbb{R}_+$, that is,$F_z(t_1, t_2, t_3, t_5, t_6) = F(t_1, t_2, t_3, z, t_5, t_6)$ is continuous at $(z, 0, 0, z, 0)$, $z \in \mathbb{R}_+$.
Continuity of F in Theorem 4.1 can be replaced by the validity of the following assumptions:

- Hypothesis (C) (or (CS)), if (B, T) satisfies property (E.A).

- Hypothesis ($\widehat{\text{C}}$) (or ($\widehat{\text{CS}}$)), if (A, S) satisfies property (E.A).
- Hypothesis (H1), if $S(X)$ or $B(X)$ is complete.
- Hypothesis (H2), if $T(X)$ or $A(X)$ is complete.

By taking $A = B$ and $S = T$ in Theorem 4.1, we obtain the following corollary:

Corollary 4.4. *Let A and S be two weakly compatible self-mappings of a metric space (X, d) suchthat*
(i) A and S satisfy property (E.A),
(ii) there exists a continuous (see Remark 4.3) function $F : \mathbb{R}+^6 \to \mathbb{R}$ in$\mathcal{F}$ such that

$$F(d(Ax, Ay),\ d(Sx, Sy),\ d(Ax, Sx),\ d(Ay, Sy),\ d(Ay, Sx),\ d(Ax, Sy)) < 0,$$

for all $x, y \in X$, where F satisfies all the conditions of implicit relation.
(iii) $A(X) \subseteq S(X)$.
Assume that one of the following conditions hold:
(iv) $\{Ay_n\}$ is a bounded sequence for every $\{y_n\} \subseteq X$ such that$\{Sy_n\}$ is convergent,
or
(v) If $\{z_n\}$, $\{r_n\}$ and $\{w_n\}$ are nonnegative sequences such that $\{z_n\} \to \infty$, $\{w_n\} \to \infty$, as $n \to \infty$ and $F(z_n, r_n, r_n, z_n, w_n, 0) \leq 0$, $n \in \mathbb{N}$, then $\{r_n\} \to \infty$, as $n \to \infty$.

If $S(X)$ or $A(X)$ is a complete subspace of X, then A and S have a unique common fixed point.

Following the lines of the proof of Theorem 4.1, we can also obtain the following result for $A = B$ and $S = T$.

Corollary 4.5. *Let A and S be two weakly compatible self-mappings of a metric space (X, d) suchthat*
(i) A and S satisfy property (E.A),
(ii) there exists a continuous function $F : \mathbb{R}+^6 \to \mathbb{R}$ in$\mathcal{F}$ such that

$$F(d(Ax, Ay),\ d(Sx, Sy),\ d(Ax, Sx),\ d(Ay, Sy),\ d(Ay, Sx),\ d(Ax, Sy)) < 0,$$

for all $x, y \in X$, where F satisfies all the conditions of implicit relation.
(iii) $A(X) \subseteq S(X)$.

If $S(X)$ or $A(X)$ is a complete subspace of X, then A and S have a unique common fixed point.

Remark 4.6. In Corollaries 4.4 and 4.5, we can take the continuous function $F(t_1, t_2, ..., t_6) = t_1 - max\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\}$, which satisfies the following

conditions:
(F_1) : $F(u,0,u,0,0,u) = u - max\{0, \frac{u}{2}, \frac{u}{2}\} = u - \frac{u}{2} = \frac{u}{2} \leq 0$ implies $u = 0$;
(F_2) : $F(u,0,0,u,u,0) = u - max\{0, \frac{u}{2}, \frac{u}{2}\} = u - \frac{u}{2} = \frac{u}{2} \leq 0$ implies $u = 0$, and
(F_u) : $F(u,u,0,0,u,u) = u - max\{u, 0, u\} = u - u = 0, \quad \forall u > 0.$
In this case, the implicit relation can be written as

$$d(Ax, Ay) < max\left\{d(Sx, Sy), \frac{d(Ax, Sx) + d(Ay, Sy)}{2}, \frac{d(Ay, Sx) + d(Ax, Sy)}{2}\right\},$$

for $x, y \in X$. Compare it with condition (ii) in Theorem 1 [1].
By setting $S = T$ in Theorem 4.1 we obtain the following corollary:
Corollary 4.7. *Let A, B and S be self-mappings of a metric space (X, d) such that*
(i) $A(X) \subseteq S(X), \quad B(X) \subseteq S(X)$,
(ii) there exists a continuous (see Remark 4.3) function $F : \mathbb{R}+^6 \to \mathbb{R}$ in $\mathcal{F}$ such that

$$F(d(Ax, By),\ d(Sx, Sy),\ d(Ax, Sx),\ d(By, Sy),\ d(By, Sx),\ d(Ax, Sy)) < 0,$$

for all $x, y \in X$, where F satisfies all the conditions of implicit relation.
(iii) (A, S) or (B, S) satisfy property (E.A),
(iv) (A, S) and (B, S) are weakly compatible.
Assume that one of the following conditions hold:
(v) $\{By_n\}$ is a bounded sequence for every $\{y_n\} \subseteq X$ such that$\{Sy_n\}$ is convergent (in case (A, S) satisfies property (E.A)), and $\{Ay_n\}$ is a bounded sequence for every $\{y_n\} \subseteq X$ such that$\{Sy_n\}$ is convergent (in case (B, S) satisfies property (E.A)),
or
(vi) If $\{z_n\}$, $\{r_n\}$ and $\{w_n\}$ are nonnegative sequences such that $\{z_n\} \to \infty$, $\{w_n\} \to \infty$, as $n \to \infty$ and

$$F(z_n, r_n, r_n, z_n, w_n, 0) \leq 0,\ n \in \mathbb{N}, \text{(in case } (A, S) \text{ satisfies (E.A))},$$

$$F(z_n, r_n, z_n, r_n, 0, w_n) \leq 0,\ n \in \mathbb{N}, \text{(in case } (B, S) \text{ satisfies (E.A))},$$

then $\{r_n\} \to \infty$, as $n \to \infty$.

If the range of one of the mappings A, B or S is a complete subspace of X, then A, B and S have a unique common fixed point.
Remark 4.8. It is clear that two non compatible self-maps also satisfy property(E.A). So our result is also true for non compatible mappings as

well.

Remark 4.9. From the proof of Theorem 4.1, we deduce that the strict '$<$' sign can be replaced by '$\leq$'if we admit that the inequality in condition (F_u) is also strict, that is, $F(u,u,0,0,u,u) > 0, \quad \forall u > 0$. The same applies to the previous corollaries.

The following remarks and examples validate our main Theorem 4.1.

Remark 4.10. If we put $F(t_1, t_2, ..., t_6) = t_1 - \psi(max\{t_2, t_4, t_5\})$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\psi(0) = 0$ and $0 < \psi(t) < t, t > 0$, then, according to Remark 4.9, the implicit relation is written as

$$F(d(Ax, By),\ d(Sx, Ty),\ d(Ax, Sx),\ d(By, Ty),\ d(By, Sx),\ d(Ax, Ty)) \leq 0$$
$$\Rightarrow d(Ax, By) - \psi(max\{d(Sx, Ty), d(By, Ty), d(By, Sx)\}) \leq 0$$
$$\Rightarrow d(Ax, By) \leq \psi(max\{d(Sx, Ty), d(By, Ty), d(By, Sx)\}).$$

For this F, we obtain
$(F_1) : F(u, 0, u, 0, 0, u) = u - \psi(0) = u \leq 0$ implies $u = 0$;
$(F_2) : F(u, 0, 0, u, u, 0) = u - \psi(u) \leq 0$ implies $u = 0$, since for$u > 0$, $u - \psi(u) > 0$, and
$(F_u) : F(u, u, 0, 0, u, u) = u - \psi(u) > 0, \quad \forall u > 0$.
Besides, condition *(vi)* of Theorem 4.1 (in case (B, T) satisfies (E.A)), (C) and (CS) hold in this example.
Indeed, to check *(vi)*, let $\{z_n\}$, $\{r_n\}$ and $\{w_n\}$ be non negative sequences such that $\{z_n\} \to \infty$, $\{w_n\} \to \infty$, as $n \to \infty$ and

$$F(z_n, r_n, z_n, r_n, 0, w_n) \leq 0,\ n \in \mathbb{N},$$

then $z_n \leq \psi(max\{r_n, r_n, 0\})$, for all n, that is, $z_n \leq \psi(r_n)$, for all n. Since $\{z_n\} \to \infty$, as $n \to \infty$, then $\{\psi(r_n)\} \to \infty$, as $n \to \infty$, which implies that $r_n > 0$ for n large enough, and $0 < \psi(r_n) < r_n$ for n large enough, hence $\{r_n\} \to \infty$, as $n \to \infty$, and *(vi)* holds for the case (B, T) satisfies (E.A).
Next, we check the validity of (CS): For ψ in the above-mentioned conditions, ψ is right upper-semicontinuous at 0. Indeed, for a given $\epsilon > 0$, there exists $\delta = \epsilon > 0$, such that

$$\psi(t) < t < \epsilon = \psi(0) + \epsilon,\ 0 < t < \delta,$$
$$\psi(0) = 0 < \psi(0) + \epsilon.$$

We check that condition (CS) holds for F if ψ is right upper-semicontinuous at 0.Let $\{p_n\}$, $\{c_n\}$ and $\{q_n\}$ be nonnegative sequences such that $\{p_n\} \to l$, $\{q_n\} \to l$, $\{c_n\} \to 0$, as $n \to \infty$, and

$$F(p_n,\ \ c_n,\ \ p_n,\ \ c_n,\ \ 0,\ \ q_n) \leq 0,\ n \in \mathbb{N},$$

that is,

$$p_n \leq \psi(max\{c_n, c_n, 0\}) = \psi(c_n),\ \forall n \in \mathbb{N}.$$

We check that $F(l,\ 0,\ l,\ 0,\ 0,\ l) \leq 0$, that is,$l \leq \psi(max\{0,0,0\}) = \psi(0)$.Note that, if $l = 0$, it is trivially satisfied.Since ψ is right upper semi-continuous at 0, given $\epsilon > 0$, there exists $\delta > 0$ such that $\psi(x) < \psi(0) + \epsilon$, for $x \in [0, \delta)$. On the other hand, using that $\{c_n\} \to 0^+$, given $\delta > 0$, there exists $\nu \in \mathbb{N}$ such that $0 \leq c_n < \delta$, for $n \geq \nu$, and hence $\psi(c_n) < \psi(0) + \epsilon$, for $n \geq \nu$,which implies $0 \leq p_n < \psi(0) + \epsilon$, for $n \geq \nu$. We have proved that $limsup_{n\to\infty} p_n \leq \psi(0)$,that is,$l \leq \psi(0)$. Hence, condition (CS) is valid.
Besides, it is easy to check the validity of condition (H1).
Therefore our main Theorem 4.1 extends Theorem B of Aamri and Moutawakil, since they considered ψ to be nondecreasing and such that $0 < \psi(t) < t$,$t > 0$, which clearly implies $\psi(0) = 0$.

Remark 4.11. Consider $F(t_1, t_2, ..., t_6) = t_1 - \psi(max\{t_2, t_4, t_6\})$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is upper semicontinuous for $t > 0$.Then conditions (C) and (CS) are valid. To check (CS), we take $\{p_n\}$, $\{c_n\}$ and $\{q_n\}$ nonnegative sequences such that $\{p_n\} \to l$, $\{q_n\} \to l$, $\{c_n\} \to 0$, as $n \to \infty$, and

$$F(p_n,\ \ c_n,\ \ p_n,\ \ c_n,\ \ 0,\ \ q_n) \leq 0,\ n \in \mathbb{N},$$

that is,

$$p_n \leq \psi(max\{c_n, c_n, q_n\}),\ \forall n \in \mathbb{N}.$$

We check that $F(l,\ 0,\ l,\ 0,\ 0,\ l) \leq 0$, that is,$l \leq \psi(max\{0,0,l\}) = \psi(l)$, or, $l \leq \psi(l)$. If $l = 0$, there is nothing to prove. Suppose that $l > 0$ and take a fixed $\epsilon > 0$. By the upper-semicontinuity of ψ at $l > 0$, we get that there exists a neighborhood $(l - \delta, l + \delta)$ of l $(\delta < l)$ such that, for every $x \in (l - \delta, l + \delta)$, then$\psi(x) < \psi(l) + \epsilon$. Since $\{max\{c_n, c_n, q_n\}\} \to l$, given $\delta > 0$, there exists $R \in \mathbb{N}$ such that $|max\{c_n, c_n, q_n\} - l| < \delta$, for every $n \geq R$. Hence $p_n \leq \psi(max\{c_n, c_n, q_n\}) < \psi(l) + \epsilon$, for $n \geq R$. Therefore, we have proved that, for every $\epsilon > 0$, there exists $R \in \mathbb{N}$ such that, for every $n \geq R$, $p_n < \psi(l) + \epsilon$.This proves that $limsup_{n\to\infty} p_n \leq \psi(l)$, thus $l \leq \psi(l)$.
A similar reasoning provides the validity of ($\widehat{\text{CS}}$) and (H1). Condition (H2) is also satisfied.
For this function F, the implicit condition can be written as

$$d(Ax, By) \leq \psi(max\{d(Sx, Ty), d(By, Ty), d(Ax, Ty)\}),$$

for x, $y \in X$,and F satisfies properties (F_1), (F_2) and (F_u) if $0 < \psi(t) < t$ for $t > 0$. Indeed,
$(F_1) : F(u, 0, u, 0, 0, u) = u - \psi(max\{0, 0, u\}) = u - \psi(u) \leq 0 \Rightarrow u = 0$,

$(F_2): F(u,0,0,u,u,0) = u - \psi(max\{0,u,0\}) = u - \psi(u) \leq 0 \Rightarrow u = 0$,
$(F_u): F(u,u,0,0,u,u) = u - \psi(max\{u,0,u\}) = u - \psi(u) > 0$, for all $u > 0$.
Example 4.12. Consider $F(t_1, t_2, ..., t_6) = \int_0^{t_1} \varphi(t)dt - \phi\left(\int_0^{max\{t_2,t_4,t_5\}} \varphi(t)dt\right)$, where$\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable mapping, summable on each compact interval,nonnegative and such that $\int_0^{\epsilon} \varphi(t)dt > 0$, $\forall \epsilon > 0$, and$\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that$\phi(0) = 0$ and$0 < \phi(t) < t$,$t > 0$.The implicit relation is written as

$$\int_0^{d(Ax,By)} \varphi(t)dt \leq \phi\left(\int_0^{max\{d(Sx,Ty),d(By,Ty),d(By,Sx)\}} \varphi(t)dt\right),$$

for $x, y \in X$. For a related problem, see Theorem 1 and Corollary 3 in [2].We can check that F satisfies the following properties:
$(F_1): F(u,0,u,0,0,u) = \int_0^u \varphi(t)dt - \phi\left(\int_0^{max\{0,0,0\}} \varphi(t)dt\right) = \int_0^u \varphi(t)dt - \phi(0) = \int_0^u \varphi(t)dt \leq 0$ implies $u = 0$;
$(F_2): F(u,0,0,u,u,0) = \int_0^u \varphi(t)dt - \phi\left(\int_0^{max\{0,u,u\}} \varphi(t)dt\right) = \int_0^u \varphi(t)dt - \phi\left(\int_0^u \varphi(t)dt\right) \leq 0$ implies $u = 0$, since for$u > 0$, $\int_0^u \varphi(t)dt > 0$ and $\int_0^u \varphi(t)dt \leq \phi\left(\int_0^u \varphi(t)dt\right) < \int_0^u \varphi(t)dt$, which is a contradiction, and
$(F_u): F(u,u,0,0,u,u) = \int_0^u \varphi(t)dt - \phi\left(\int_0^{max\{u,0,u\}} \varphi(t)dt\right) = \int_0^u \varphi(t)dt - \phi\left(\int_0^u \varphi(t)dt\right) > 0, \quad \forall u > 0$.
Besides, condition *(vi)* of Theorem 4.1 (for the case (B,T) satisfies (E.A)), (C) and (CS) hold in this example.
Let $\{z_n\}$, $\{r_n\}$ and $\{w_n\}$ be nonnegative sequences such that $\{z_n\} \to \infty$, $\{w_n\} \to \infty$, as $n \to \infty$ and

$$F(z_n, r_n, z_n, r_n, 0, w_n) \leq 0, \ n \in \mathbb{N},$$

then

$$\int_0^{z_n} \varphi(t)dt \leq \phi\left(\int_0^{max\{r_n,r_n,0\}} \varphi(t)dt\right) = \phi\left(\int_0^{r_n} \varphi(t)dt\right), \ n \in \mathbb{N}.$$

Since $\{z_n\} \to \infty$, as $n \to \infty$, then $\int_0^{z_n} \varphi(t)dt > 0$, for $n \geq n_0$, and hence $\phi\left(\int_0^{r_n} \varphi(t)dt\right) > 0$, for$n \geq n_0$, which implies $\int_0^{r_n} \varphi(t)dt > 0$, for $n \geq n_0$, and $\int_0^{z_n} \varphi(t)dt < \int_0^{r_n} \varphi(t)dt$, $n \geq n_0$. Since $\int_0^u \varphi(t)dt$ is nondecreasing in$u$, then $z_n < r_n$, $n \geq n_0$, which joint to $\{z_n\} \to \infty$ implies$\{r_n\} \to \infty$, as $n \to \infty$.
To check the validity of (CS),take $\{p_n\}$, $\{c_n\}$ and $\{q_n\}$ nonnegative sequences such that $\{p_n\} \to l$, $\{q_n\} \to l$, $\{c_n\} \to 0$, as $n \to \infty$, and

$$F(p_n, \ c_n, \ p_n, \ c_n, \ 0, \ q_n) \leq 0, \ n \in \mathbb{N},$$

which implies

$$\int_0^{p_n} \varphi(t)dt \le \phi\left(\int_0^{max\{c_n,c_n,0\}} \varphi(t)dt\right) = \phi\left(\int_0^{c_n} \varphi(t)dt\right),\ n \in \mathbb{N}.$$

We want to prove that $F\,(l,\ \ 0,\ \ l,\ \ 0,\ \ 0,\ \ l) \le 0$, that is,

$$\int_0^{l} \varphi(t)dt \le \phi\left(\int_0^{max\{0,0,0\}} \varphi(t)dt\right) = \phi\,(0) = 0,$$

or, $l = 0$.From the implicit relation and the hypotheses on ψ, we obtain, for n with $c_n = 0$,

$$\int_0^{p_n} \varphi(t)dt \le \phi\,(0) = 0,$$

and for n with $c_n > 0$,

$$\int_0^{p_n} \varphi(t)dt \le \phi\left(\int_0^{c_n} \varphi(t)dt\right) < \int_0^{c_n} \varphi(t)dt.$$

From these inequalities, and the fact that $\{c_n\} \to 0$, we deduce that $\int_0^l \varphi(t)dt \le 0$, hence $l = 0$, and condition (CS) follows.

Similarly, the validity of (H1) can be deduced.

Example 4.13. Let A, B, S and T be four self-maps on $X = [2, \infty)$, with the usual metric $d(x, y) = |x - y|$, defined by:

$Ax = 2,\quad Tx = x,\quad Bx = 2,\ \ \forall\ x \in X$, and $Sx = 2$ if x is a rational number of X, and $Sx = 3$ if x is an irrational number of X.

Let us define a function $F \in \mathcal{F}$ such that $F : {\mathbb{R}_+}^6 \to \mathbb{R}$ where $F(t_1, ..., t_6) = t_1 - h\,max\{2t_2, t_3, t_4, t_5, t_6\}$, for each $t_1, ..., t_6 \ge 0$, where $0 \le h < \frac{1}{2}$, then we observe that:

(i) $A(X) = \{2\} \subseteq T(X) = X$ and $B(X) = \{2\} \subseteq S(X) = \{2, 3\}$.

(ii) Let us discuss two cases for the elements $x, y \in X$ and obtain the implicit relation:

Case I. If x is a rational number of X and $y \in X$, then we have:

$F(0,\ |2-y|,\ 0,\ |2-y|,\ 0,\ |2-y|) = 0 - h\,max\{2|2-y|,\ 0,\ |2-y|,\ 0,\ |2-y|\}$ $= -2h|2-y| \le 0,$ as $0 \le h < \frac{1}{2}$.

Case II. If x is an irrational number of X and $y \in X$, we have:

$$F(0,\ |3-y|,\ 1,\ |y-2|,\ 1,\ |y-2|) = 0 - h\,max\{2|3-y|,\ 1,\ |y-2|,\ 1,\ |y-2|\} \le 0,$$

as $0 \le h < \frac{1}{2}$.

Moreover,

$(F_1): F(u,0,u,0,0,u) = u - h\,max\{0,u,0,0,u\} = u(1-h) \leq 0 \Rightarrow u = 0,$
$(F_2): F(u,0,0,u,u,0) = u - h\,max\{0,0,u,u,0\} = u(1-h) \leq 0 \Rightarrow u = 0,$
$(F_u): F(u,u,0,0,u,u) = u - h\,max\{2u,0,0,u,u\} = u(1-2h) > 0,$
for all $u > 0$, as $0 \leq h < \frac{1}{2}$; showing that (F_1), (F_2) and (F_u) are satisfied.
(iii) Both (A,S) and (B,T) are weakly compatible. Indeed, $Ax = Sx$ if and only if $x \in \mathbb{Q} \cap X$,and, for these points, $ASx = A(2) = 2 = S(2) = SAx$. On the other hand, $Bx = Tx$ if and only if $x = 2$ and $BT(2) = B(2) = 2 = T(2) = TB(2)$.
(iv) Let us discuss the property (E.A) for a given pair of mappings. Suppose that $\{y_n\}$ is an arbitrary sequence in X, then we have $Ay_n = 2$, $Sy_n = 2$ if y_n is a rational number, and $Sy_n = 3$ if y_n is an irrational number.Therefore $lim_n Ay_n = lim_n Sy_n = 2$, iff y_n is a rational number of X (for n large). For instance, consider the sequence $\{y_n\} = \left\{2 + \frac{1}{n}\right\}$. Thus $lim_n Ay_n = lim_n Sy_n = 2$. Hence the pair$(A,S)$ satisfies property (E.A).

On the other hand, if we take a sequence $\{y_n\}$ of irrational numbers of X (e.g., a subsequence of $y_n = 2 + \frac{1}{\sqrt{n}}$ corresponding to $n \neq k^2$, $k \in \mathbb{N}$), then$lim_n Ay_n = 2 \neq lim_n Sy_n = 3$ and such a sequence is not appropriate to check the validity of property (E.A).
Besides, for any sequence in X with $\{y_n\} \to 2$ (e.g. $\{y_n\} = \left\{2 + \frac{1}{n}\right\}$), we obtain $lim_n By_n = lim_n Ty_n = 2$, and (B,T) satisfies property (E.A). Thus both pairs satisfy property (E.A) and condition *(iv)* is satisfied for both pairs.
(v) For every $\{y_n\} \subseteq X$ such that$\{Ty_n\} = \{y_n\}$ is convergent, then $\{By_n\} = \{2\}$ is a bounded sequence ((A,S) satisfies property (E.A)), and, besides, for every $\{y_n\} \subseteq X$ such that$\{Sy_n\}$ is convergent, then$\{Ay_n\} = \{2\}$ is a bounded sequence ((B,T) satisfies property (E.A)).
(vi) The subset $T(X) = X$ is complete.

Hence all the conditions of our theorem are satisfied and $x = 2$ is the only common fixed point of A, B, S and T. This validates our main Theorem.
Example 4.14. Let A, B, S and T be the same four self-maps of the previous Example on $X = [2,\infty)$, with the usual metric$d(x,y) = |x - y|$. Consider function $F \in \mathcal{F}$ such that $F : {\mathbb{R}_+}^6 \to \mathbb{R}$ where $F(t_1,...,t_6) = t_1 - h\,max\{2t_2, t_4 - t_3, t_5 - t_6\}$, for each $t_1,...,t_6 \geq 0$, where $0 < h < \frac{1}{2}$, then Fsatisfies that:
$(F_1): F(u,0,u,0,0,u) = u - h\,max\{0,-u,-u\} = u \leq 0 \Rightarrow u = 0,$
$(F_2): F(u,0,0,u,u,0) = u - h\,max\{0,u,u\} = u(1-h) \leq 0 \Rightarrow u = 0,$
$(F_u): F(u,u,0,0,u,u) = u - h\,max\{2u,0,0\} = u(1-2h) > 0$,for all $u > 0$.
Moreover, for $x, y \in X$:

Case I. If x is a rational number of X and $y \in X$, then:
$F(0,\ |2-y|,\ 0,\ |2-y|,\ 0,\ |2-y|) = -h\,max\{2|2-y|,\ |2-y|,\ \ -|2-y|\}$
$= -2h|2-y| \leq 0.$
Case II. If x is an irrational number of X and $y \in X$, we have:

$$F(0,\ |3-y|,\ 1,\ |y-2|,\ 1,\ |y-2|) = 0-h\,max\{2|3-y|,\ |y-2|-1,\ 1-|y-2|\} \leq 0.$$

In this case, F satisfies property *(vi)* in Theorem 4.1 (note that (B,T) satisfies property (E.A)). Indeed, suppose that $\{z_n\}$, $\{r_n\}$ and $\{w_n\}$ are nonnegative sequences such that $\{z_n\} \to \infty$, $\{w_n\} \to \infty$, as $n \to \infty$, and

$$F(z_n, r_n, z_n, r_n, 0, w_n) = z_n - h\,max\{2r_n, r_n - z_n, -w_n\} \leq 0,\ n \in \mathbb{N}.$$

Hence, taking into account the sign of the sequences, we get

$$z_n \leq h\,max\{2r_n, r_n - z_n, -w_n\} = 2hr_n,$$

for all n, and in consequence, $\{z_n\} \to \infty$ implies $\{r_n\} \to \infty$, as $n \to \infty$.
See Example 4.13, for the validity of other hypotheses in Theorem 4.1.
Remark 4.15. If we take $S = id_X$ the identity map in Corollary 4.7, then $A(X) \subseteq id_X(X)$, and $B(X) \subseteq id_X(X)$ is trivially valid and the implicit relation can be written as

$$F(d(Ax, By),\ d(x, y),\ d(Ax, x),\ d(By, y),\ d(By, x),\ d(Ax, y)) \leq 0,$$

for all $x, y \in X$, where F must satisfy all the conditions of implicit relation (see [8]). We have to impose that (A, id_X) or(B, id_X) satisfy property (E.A), since the range of id_X is complete in the sense of Theorem 4.1 and (A, id_X) and (B, id_X) are weakly compatible (they are commuting pairs). Adding the validity of one of the properties

- $\{By_n\}$ is a bounded sequence for every convergent sequence $\{y_n\} \subseteq X$ (in case (A, id_X) satisfies property (E.A)), and $\{Ay_n\}$ is a bounded sequence for every convergent sequence $\{y_n\} \subseteq X$ (in case (B, id_X) satisfies property(E.A)),

- or condition *(vi)* in Corollary 4.7 with $S = id_X$.

we deduce the existence of a unique common fixed point for A, B.
Consider $F(t_1, t_2, ..., t_6) = G(t_1) - \psi(G(max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}))$, where $G : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous for$t > 0$ and such that$G(0) = 0$, $G(t) > 0$, for $t > 0$, and$\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is upper semicontinuous for $t > 0$ and such

that $\psi(0) = 0$ and $\psi(t) < t$, for $t > 0$.In this case, the implicit condition in Theorem 4.1 (see Remark 4.9) can be written as

$$G(d(Ax, By)) \\ \leq \psi(G(max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{1}{2}(d(By, Sx)+d(Ax, Ty))\})),$$

for all $x, y \in X$.
If $S = T = id_X$, then we get

$$G(d(Ax, By)) \\ \leq \psi(G(max\{d(x, y), d(Ax, x), d(By, y), \frac{1}{2}(d(By, x) + d(Ax, y))\})),$$

for all $x, y \in X$, which is similar to condition in Theorem 1 [8].
We check that the above defined function F satisfies the conditions of implicit relation:
$(F_1) : F(u, 0, u, 0, 0, u) = G(u)-\psi(G(max\{0, u, 0, \frac{1}{2}u\}) = G(u)-\psi(G(u)) \leq 0 \Rightarrow u = 0$,since, if $u > 0$, then $G(u) > 0$ and $G(u) \leq \psi(G(u)) < G(u)$, which is a contradiction.
$(F_2) : F(u, 0, 0, u, u, 0) = G(u)-\psi(G(max\{0, 0, u, \frac{1}{2}u\}) = G(u)-\psi(G(u)) \leq 0 \Rightarrow u = 0$,
$(F_u) : F(u, u, 0, 0, u, u) = G(u) - \psi(G(max\{u, 0, 0, u\}) = G(u) - \psi(G(u)) > 0$, for all $u > 0$.
As a particular case, we can consider $G(u) = \int_0^u \varphi(s)\, ds$, where $\varphi \geq 0$ is Lebesgue-integrable and such that $\int_0^\epsilon \varphi(t)dt > 0$, for every $\epsilon > 0$, and $\psi(t) = kt$, with $0 \leq k < 1$ (see [8]).Note that conditions (C) and (CS) are valid for F. Next, we check the validity of (CS).Indeed, for $\{p_n\}$, $\{c_n\}$ and $\{q_n\}$ nonnegative sequences such that $\{p_n\} \to l$, $\{q_n\} \to l$, $\{c_n\} \to 0$, as $n \to \infty$, and

$$F(p_n, \quad c_n, \quad p_n, \quad c_n, \quad 0, \quad q_n) \leq 0,\ n \in \mathbb{N},$$

we get

$$G(p_n) \leq \psi\left(G\left(max\left\{c_n, p_n, c_n, \frac{q_n}{2}\right\}\right)\right),\ \forall n \in \mathbb{N}.$$

We have to check that $F\,(l, \quad 0, \quad l, \quad 0, \quad 0, \quad l) \leq 0$, that is,

$$G(l) \leq \psi\left(G\left(max\left\{0, l, 0, \frac{1}{2}l\right\}\right)\right) = \psi\left(G\left(l\right)\right).$$

If $l = 0$, this inequality is reduced to $0 = G(0) \leq \psi\left(G\left(0\right)\right) = \psi(0)$, which is obviously valid.

Suppose that $l > 0$. By the upper-semicontinuity of ψ at $G(l) > 0$, given $\epsilon > 0$, there exists a neighborhood $(G(l)-\delta, G(l)+\delta)$ of $G(l)$ such that, for every $x \in (G(l)-\delta, G(l)+\delta)$, $\psi(x) < \psi(G(l)) + \epsilon$. Since G is continuous at $l > 0$, then $\{G\left(max\left\{c_n, p_n, c_n, \frac{q_n}{2}\right\}\right)\} \to G(l)$, as $n \to \infty$, therefore, given $\delta > 0$, there exists $R \in \mathbb{N}$ such that, for $n \geq R$, $G\left(max\left\{c_n, p_n, c_n, \frac{q_n}{2}\right\}\right) \in (G(l) - \delta, G(l)+\delta)$, and $\psi\left(G\left(max\left\{c_n, p_n, c_n, \frac{q_n}{2}\right\}\right)\right) < \psi(G(l)) + \epsilon$. Therefore,

$$G(p_n) \leq \psi\left(G\left(max\left\{c_n, p_n, c_n, \frac{q_n}{2}\right\}\right)\right) < \psi(G(l)) + \epsilon,\ n \geq R.$$

This proves that $limsup_{n\to\infty} G(p_n) \leq \psi(G(l))$, that is, $G(l) \leq \psi(G(l))$, and condition (CS) is valid.

Similarly, we obtain the validity of $(\widehat{\mathrm{CS}})$. It is also easy to check that (H1) and (H2) are satisfied.

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