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INFINITESIMAL DEFORMATIONS OF BASIC TENSOR IN GENERALIZED RIEMANNIAN SPACE

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Abstract. At the beginning of the present work the basic facts on generalized Riemannian space (GR_N) in the sense of Eisenhart's definition [Eis] and also on infinitesimal deformations of a space are given. We study the Lie derivatives and infinitesimal deformations of basic covariant and contravariant tensor at GR_N.

1. Introduction

1.0. At the beginning we are giving basic information on generalized Riemannian spaces and on infinitesimal deformations of a space.

1.1. A generalized Riemannian space GR_N at the sense of Eisenhart's definition [Eis] is a differentiable manifold, endowed with nonsymmetric basic tensor $g_{ij}(x^1,\dots,x^N)$, where x^i are local coordinates. So, generally we have

$$g_{ij}(x) \neq g_{ji}(x). \tag{1.1}$$

The symmetric, respectively antisymmetric part of the basic tensor are

$$h_{ij} = \frac{1}{2}(g_{ij}+g_{ji}), \quad k_{ij} = \frac{1}{2}(g_{ij}-g_{ji}), \tag{1.2 a,b}$$

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from where it follows that

$$g_{ij} = h_{ij} + k_{ij}. \tag{1.3}$$

For the lowering and raising of indices in GR_N one uses the tensor h_{ij} respectively h^{ij}, where

$$(h^{ij}) = (h_{ij})^{-1} \qquad (det(h_{ij}) \neq 0). \tag{1.4}$$

Christoffel symbols at GR_N are

$$\Gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}), \quad \Gamma^i_{jk} = h^{ip}\Gamma_{p.jk}, \tag{1.5 a,b}$$

where, for example, $g_{ji,k} = \partial g_{ji}/\partial x^k$. Based on (1.1) the non-symmetry of Christoffel symbols with respect to j, k at (1.5) follows. The symbols Γ^i_{jk} are connection coefficients at GR_N.

By a reason of non-symmetry of the connection, one can use in GR_N four kinds of covariant derivatives of a tensor. For example:

$$t^i_{j\underset{\substack{1\\2\\3\\4}}{|}m} = t^i_{j,m} + \Gamma^i_{\underset{\substack{mp\\pm\\mp}}{pm}} t^p_j - \Gamma^p_{\underset{\substack{mj\\mj\\jm}}{jm}} t^i_p \tag{1.6$a-d$}$$

1.2. Basic facts on infinitesimal deformations and their expressions by Lie derivative one can find, e.g., at [Ya,Ya1,St,Sch,MVS,MVS1].

Definition 1.1. A transformation $f : GR_N \to GR_N : x = (x^1, \ldots, x^N) \equiv (x^i) \to \bar{x} = (\bar{x}^1, \ldots, \bar{x}^N) \equiv (\bar{x}^i)$, where

$$\bar{x}^i = x^i + z^i(x^j)\varepsilon, \quad i, j = 1, \ldots, N, \tag{1.7}$$

ε being an infinitesimal, is called **infinitesimal deformation of a space** GR_N, determined by the vector field $z = (z^i)$, which is called a **field of infinitesimal deformations** (1.7).

We denote with (i) local coordinate system in which the point x is endowed with coordinates x^i, and the point $\bar{x}$ with the coordinates $\bar{x}^i$. We will also introduce **a new coordinate system** (i'), corresponding to the point $x = (x^i)$ new coordinates

$$x^{i'} = \bar{x}^i, \tag{1.8}$$

i.e. as new coordinates $x^{i'}$ of the point $x = (x^i)$ we choose old coordinates (at the system (i)) of the point $\bar{x} = (\bar{x}^i)$. Namely, at the system (i') is $x = (x^{i'}) \underset{(1.8)}{=} (\bar{x}^i)$, where $\underset{(1.8)}{=}$ denotes "equal according to (1.8)".

Let us consider a geometric object $\mathcal{A}$ with respect to the system (i) at the point $x = (x^i) \in GR_N$, denoting this with $\mathcal{A}(i, x)$.

Definition 1.2. The point $\bar{x}$ is said to be **deformed point** of the point x, if (1.7) holds. Geometric object $\bar{\mathcal{A}}(i, x)$ is **deformed object** $\mathcal{A}(i, x)$ with respect to deformation (1.7), if its value at system (i'), at the point x is equal to the value of the object $\mathcal{A}$ at the system (i) at the point $\bar{x}$, i.e. if

$$\bar{\mathcal{A}}(i', x) = \mathcal{A}(i, \bar{x}). \tag{1.9}$$

Definition 1.3. The magnitude $\mathcal{DA}$, the difference between deformed object $\bar{\mathcal{A}}$ and initial object $\mathcal{A}$ at the same coordinate system and at the same point with respect to (1.7), i.e.

$$\mathcal{DA} = \bar{\mathcal{A}}(i, x) - \mathcal{A}(i, x), \tag{1.10}$$

is called **Lie difference (Lie differential)**, and the magnitude

$$\mathcal{L}_z\mathcal{A} = \lim_{\varepsilon \to 0} \frac{\mathcal{DA}}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\bar{\mathcal{A}}(i, x) - \mathcal{A}(i, x)}{\varepsilon} \tag{1.11}$$

is **Lie derivative** of geometric object $\mathcal{A}(i, x)$ with respect to the vector field $z = (z^i(x^j))$.

Using the relation (1.10), for deformed object $\bar{\mathcal{A}}(i, x)$ we have

$$\bar{\mathcal{A}}(i, x) = \mathcal{A}(i, x) + \mathcal{DA}, \tag{1.12}$$

and thus we can express $\bar{\mathcal{A}}$, finding previously $\mathcal{DA}$. The known main cases are:

According to (1.10) we have $\mathcal{D}x^i = \bar{x}^i - x^i$, i.e. for the **coordinates** we have

$$\mathcal{D}x^i = z^i(x^j)\varepsilon, \tag{1.13}$$

from where

$$\mathcal{L}_z x^i = z^i(x^j). \tag{1.13'}$$

For the **scalar function** $\varphi(x) \equiv \varphi(x^1, \dots, x^N)$ we have

$$\mathcal{D}\varphi(x) = \varphi_{,p} z^p(x)\varepsilon = \mathcal{L}_z \varphi(x)\varepsilon, \quad (\varphi_{,p} = \partial\varphi/\partial x^p), \tag{1.14}$$

i.e. Lie derivative of the scalar function is derivative of this function in direction of the vector field z.

For a **tensor of the kind** (u, v) we get

$$\begin{aligned} \mathcal{D}t^{i_1\dots i_u}_{j_1\dots j_v} &= [t^{i_1\dots i_u}_{j_1\dots j_v,p} z^p - \sum_{\alpha=1}^{u} z^{i_\alpha}_{,p} \binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} + \sum_{\beta=1}^{v} z^p_{,j_\beta} \binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v}]\varepsilon \\ &= \mathcal{L}_z t^{i_1\dots i_u}_{j_1\dots j_v}\varepsilon, \end{aligned} \tag{1.15}$$

where we denoted

$$\binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} = t^{i_1\dots i_{\alpha-1} p i_{\alpha+1}\dots i_u}_{j_1\dots j_v}, \qquad \binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v} = t^{i_1\dots i_u}_{j_1\dots j_{\beta-1} p j_{\beta+1}\dots j_v}. \tag{1.16}$$

For the **vector** dx^i we have

$$\mathcal{D}(dx^i) = \mathcal{L}_z(dx^i) = 0. \tag{1.17}$$

In the same way, as for the tensors, for the **connection coefficients** we have

$$\mathcal{D}L^i_{jk} = (L^i_{jk,p} z^p + z^i_{,jk} - z^i_{,p} L^p_{jk} + z^p_{,j} L^i_{pk} + z^p_{,k} L^i_{jp})\varepsilon = \mathcal{L}_z L^i_{jk}\varepsilon. \tag{1.18}$$

From (1.11) we have

$$\mathcal{D}\mathcal{A} = \varepsilon\mathcal{L}_z\mathcal{A}, \tag{1.19}$$

and, for study infinitesimal deformations of geometric objects, it is enough to study their Lie derivatives, and that is what we are doing in the present work.

2. Lie derivative of the basic tensor

2.1. Based on the equations (2.12) at [VMS], for the Lie derivative of a tensor $t^{i_1\dots i_u}_{j_1\dots j_v}$ we have

$$\mathcal{L}_z t^{i_1\dots i_u}_{j_1\dots j_v} = t^{i_1\dots i_u}_{j_1\dots j_v \underset{\lambda}{|} p} z^p - \sum_{\alpha=1}^{u} z^{i_\alpha}_{\underset{\mu}{|} p} \binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} + \sum_{\beta=1}^{v} z^{p}_{\underset{\nu}{|} j_\beta} \binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v}, \tag{2.1}$$

where $(\lambda,\mu,\nu) \in \{(1,2,2),(2,1,1),(3,4,3),(4,3,4)\}$. By applying to the tensor g_{ij} one obtains two cases

$$\mathcal{L}_z g_{ij} = g_{ij\underset{1}{|}p} z^p + z^p_{\underset{2}{|}i} g_{pj} + z^p_{\underset{2}{|}j} g_{ip},$$
$$\mathcal{L}_z g_{ij} = g_{ij\underset{2}{|}p} z^p + z^p_{\underset{1}{|}i} g_{pj} + z^p_{\underset{1}{|}j} g_{ip},$$

because the third case reduces to the second and the fourth case reduces to the first case. On account of

$$h_{ij\underset{\theta}{|}p} = 0, \quad \theta = 1,2, \tag{2.2}$$

the previous equations become

$$\mathcal{L}_z g_{ij} = k_{ij\underset{1}{|}p} z^p + z_{i\underset{2}{|}j} + z_{j\underset{2}{|}i} + z^p_{\underset{2}{|}i} k_{pj} + z^p_{\underset{2}{|}j} k_{ip}, \tag{2.3a}$$

$$\mathcal{L}_z g_{ij} = k_{ij\underset{2}{|}p} z^p + z_{i\underset{1}{|}j} + z_{j\underset{1}{|}i} + z^p_{\underset{1}{|}i} k_{pj} + z^p_{\underset{1}{|}j} k_{ip}. \tag{2.3b}$$

In the case of Riemannian space R_N ($g_{ij} = g_{ji} = h_{ij}, \quad k_{ij} = 0$), we obtain the known equation

$$\mathcal{L}_z g_{ij} = z_{i;j} + z_{j;i}, \tag{2.4}$$

where by ; is denoted covariant derivative with respect to Christoffel symbols at R_N. From here it follows that, forming at GR_N by h_{ij} (symmetric part of g_{ij}) Christoffel symbols $\underset{0}{\Gamma}_{i.jk}, \quad \underset{0}{\Gamma}^i_{jk}$, with respect to (1.5), we obtain a corresponding R_N, and, based on (2.4):

$$\mathcal{L}_z h_{ij} = z_{i;j} + z_{j;i}, \tag{2.5}$$

Based on exposed the following theorem is valid.

Theorem 2.1. *If k_{ij} is antisymmetric part of the basic tensor g_{ij} of the space GR_N and the covariant derivatives are defined by virtue of* (1.6)*, then for Lie derivatives are in the force equations* (2.3)*, where $z(x^i)$ is the infinitesimal deformations vector field* (1.7)*. For the symmetric part h_{ij} of the basic tensor is valid* (2.5)*, where the covariant derivative is defined by virtue of h_{ij}.*

2.2. We shall examine now the Lie derivative of the tensor g^{ij}, where

$$g^{ij} = h^{ip} h^{jq} g_{pq}. \tag{2.6}$$

Let us firstly determine $\mathcal{L}_z h^{ij}$. From (1.4) is

$$h^{ip} h_{pj} = \delta^i_j, \tag{2.7}$$

herefrom, using properties of the Lie derivatives:

$$(\mathcal{L}_z h^{ip}) h_{pj} + h^{ip} \mathcal{L}_z h_{pj} = 0 \Rightarrow (\mathcal{L}_z h^{ip}) h_{pj} = -h^{ip} (\mathcal{L}_z h_{pj}).$$

From here, composing with h^{jq} and applying (2.7):

$$\mathcal{L}_z h^{ij} = -h^{ip} h^{jq} (\mathcal{L}_z h_{pq}). \tag{2.8}$$

For the non symmetric tensor g^{ij} based on (2.6), we obtain

$$\begin{aligned} \mathcal{L}_z g^{ij} &= (\mathcal{L}_z h^{ip}) h^{jq} g_{pq} + h^{ip} (\mathcal{L}_z h^{jq}) g_{pq} + h^{ip} h^{jq} (\mathcal{L}_z g_{pq}) \\ &\underset{(2.8)}{=} - h^{ir} h^{ps} (\mathcal{L}_z h_{rs}) h^{jq} g_{pq} - h^{ip} h^{jr} h^{qs} (\mathcal{L}_z h_{rs}) g_{pq} + h^{ip} h^{jq} (\mathcal{L}_z g_{pq}) \\ &= - h^{ir} (\mathcal{L}_z h_{rs}) g^{sj} - h^{jr} g^{is} (\mathcal{L}_z h_{rs}) + h^{ip} h^{jq} (\mathcal{L}_z g_{pq}), \end{aligned}$$

where e.g., $\underset{(2.8)}{=}$ signifies: equal on the base of (2.8). Finally, the previous equation can be written in the form:

$$\mathcal{L}_z g^{ij} = -(h^{ip} g^{qj} + h^{jp} g^{iq}) \mathcal{L}_z h_{pq} + h^{ip} h^{jq} \mathcal{L}_z g_{pq}. \tag{2.9}$$

Thus, we have

Theorem 2.2. *The Lie derivative of the tensor g^{ij}, defined with* (2.6)*, is given by the equation* (2.9)*, where h_{ij} is the symmetric part of the basic tensor g_{ij} of the generalized Riemannian space GR_N, and h^{ij} is defined by*

(1.4). *For a symmetric* $g_{ij}(g_{ij} = h_{ij} = h_{ji})$, *the equation* (2.9) *reduce to* (2.8).

2.3. We shall express $\mathcal{L}_z g_{ij}$ and $\mathcal{L}_z g^{ij}$ by virtue of covariant derivatives formed with respect of $\underset{0}{\Gamma}^i_{jk}$. Summing the equations (2.3), we get

$$2\mathcal{L}_z g_{ij} = (k_{ij\underset{1}{|}p} + k_{ij\underset{2}{|}p})z^p + (z_{i\underset{1}{|}j} + z_{i\underset{2}{|}j}) + (z_{j\underset{1}{|}i} + z_{j\underset{2}{|}i})$$
$$+ (z^p_{\underset{1}{|}i} + z^p_{\underset{2}{|}i})k_{pj} + (z^p_{\underset{1}{|}j} + z^p_{\underset{2}{|}j})k_{ip}.$$

Summing covariant derivatives of the first and the second kind, one obtains a covariant derivative of the same tensor in relation to the symmetric connection $\underset{0}{\Gamma}^i_{jk}$. So, from the previous equation we obtain the equation

$$\mathcal{L}_z g_{ij} = k_{ij;p}z^p + z_{i;j} + z_{j;i} + z^p_{;i}k_{pj} + z^p_{;j}k_{ip}, \tag{2.10}$$

which is of the form (2.3).

By substitution at (2.9) with respect to (2.5) and (2.10), or by direct use of (2.1), we obtain

$$\mathcal{L}_z g^{ij} = k^{ij}_{;p}z^p - z^i_{;p}g^{pj} - z^j_{;p}g^{ip}. \tag{2.11}$$

Theorem 2.3. *Using covariant derivatives with respect to symmetric part* $\underset{0}{\Gamma}^i_{jk}$ *of the second kind Cristoffel symbols, Lie derivative of the basic tensor* g_{ij} *is given by* (2.10), *and the Lie derivative of the tensor* g^{ij}, *defined in* (2.6), *is given by* (2.11), *where* k_{ij} *is antisymmetric part of* g_{ij}.

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