Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **21:2** (2007), 243–249

QUASI-NEARLY SUBHARMONIC FUNCTIONS AND CONFORMAL MAPPINGS

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Abstract

If φ is a conformal mapping and u is a quasi-nearly subharmonic function, then $u \circ \varphi$ is quasi-nearly subharmonic. A similar fact for "regularly oscillating" functions holds.

Introduction

If u is a nonnegative subharmonic function on a domain $\Omega \subset \mathbb{C}$ and $p \geq 1$, then

$$u(z)^p \leq \frac{1}{r^2} \int_{B(z,r)} u^p \, dm \tag{1}$$

for all $B(x,r) \subset \Omega$. Here $B(z,r)$ is the Euclidean disk with center z and radius r, and dm is the Lebesgue measure in $\mathbb{C}$ normalized so that the measure of the unit ball equals one. If $0 < p < 1$, then (1) need not hold but there is a constant $C = C(p) \geq 1$ such that

$$u(z)^p \leq \frac{C}{r^2} \int_{B(z,r)} u^p \, dm \tag{2}$$

This fact, essentially due to Hardy and Littlewood [5], was first proved by Fefferman and Stein [3, Lemma 2, p. 172]. Fefferman and Stein's proof is reproduced in Garnett [4, Lemma 3.7, pp. 121–123]. Although Fefferman

2000 *Mathematics Subject Classification.* Primary: 31B05, 31B25; Secondary: 31C05.
Key words and phrases. Subharmonic, quasi-nearly subharmonic, and regularly oscillating functions, Koebe's theorems.
Received: August 20, 2007
The author is supported by MNZŽS Grant ON144010, Serbia

and Stein considered only the case when $u = |v|$ and v is harmonic their proof applies also in the case of nonnegative subharmonic functions. Perhaps, the first simple proof of (2) was given in [10, p. 64], although it depended on the hypothesis that u was subharmonic. For other proofs see [13, p. 18, and Theorem 1, p. 19], (see also [14, Theorem A, p. 15]), [18, Lemma 2.1, p. 233], and [19, Theorem, p. 188]).

That the validity of (2) for some p implies its validity for all p was observed in [1, p. 132], [13, Theorem 1], and [18, Lemma 2.1]. See also [11], where the case of "M-subharmonic" functions was considered, and [8, 9], where some extensions of [11] were made.

For various applications of (2) we refer to [24, p. 191] , [6, Theorems 1 and 2, pp. 117-118], [21, Theorems 1, 2 and 3, pp. 301, 307], [18, Theorem, p. 233], [22, Theorem 2, p. 271], [23, Theorem, p. 113], [15], [12], and [17]. Further information can be found in [19] and [17].

Quasi-nearly subharmonic functions

Let Ω be a subdomain of the complex plane $\mathbb{C}$. Following [20] and [17], we call a Borel measurable function $u : \Omega \to [0, \infty]$ *quasi-nearly subharmonic*, if $u \in L^1_{\mathrm{loc}}(\Omega)$ and if there is a constant $K = K(u, \Omega) \geq 1$ such that

$$u(z) \leq \frac{K}{r^2} \int_{B(z,r)} u(w)\, dm(w) \tag{3}$$

for any disk $B(z, r) \subset \Omega$.

In [18], the term *pseudoharmonic functions* is used, while in [13], condition (3) is called sh_K-condition. If $K = 1$ (and u takes its values in $[-\infty, \infty]$), then u is called *nearly subharmonic* (see [7]).

We will denote by $QNS_K(\Omega)$ the class of all functions satisfying (3) (for a fixed K) and by $QNS(\Omega)$ the class of all quasi-nearly subharmonic functions defined in Ω; so

$$QNS(\Omega) = \bigcup_{K \geq 1} QNS_K(\Omega).$$

One of the most important properties of QNS is the following fact, which generalizes the above mentioned result of Fefferman and Stein [3].

Theorem A. *[1, 13, 18] If $u \in QNS_K(\Omega)$, and $p > 0$, then $u^p \in QNS_C(\Omega)$, where C is a constant depending only on p, K. In particular, if u^p is quasi-nearly subharmonic for some $p > 0$, then so is for every $p > 0$.*

In this paper we prove the following:

Theorem 1. *Let $u \in QNS_K(\Omega)$ and φ a conformal mapping from a domain G onto Ω, then the composition $u \circ \varphi$ belongs $QNS_C(G)$, where C depends only on K.*

Regularly oscillating functions

A function f defined in Ω is called regularly oscillating (see [17]) if f is of class $C^1(\Omega)$ and

$$|\nabla f(z)| \leq Kr^{-1} \sup_{B(z,r)} |f - f(z)|, \qquad B(z,r) \subset \Omega. \tag{4}$$

The class of such functions is denoted in [13] and [16] by $OC^1_K(\Omega)$ ($O =$ oscillation). The class of all regularly oscillating functions will be denoted by $RO(\Omega)$.

Theorem B. *[13, Theorem 3] If f is regularly oscillating, then $|f|$ and $|\nabla f|$ are quasi-nearly subharmonic. Moreover, if $f \in OC^1_K(\Omega)$, then $|f|$ and $|\nabla f|$ are in $QNS_C(\Omega)$, where C depends only on K.*

Example 1. Harmonic functions are regularly oscillating.

Example 2. [13] Convex functions are regularly oscillating. It follows that the modulus of the gradient of a convex function is quasi-nearly subharmonic.

Example 3. [14] If f is an eigenfunction of Δ, i.e., $\Delta f = \lambda f$ for some constant λ, and if Ω is bounded, then $f \in OC^2(\Omega)$.

Example 4 (**Polyharmonic functions**). A function $f \in C^{2k}(\Omega)$ is said to be polyharmonic (of degree k) if it is annihilated by the k-th power of the Laplacian. It is proved in [14, Corollary 5] (see also [15]) that every polyharmonic function is regularly oscillating, and therefore $|f|$ and $|\nabla f|$ are quasi-nearly subharmonic.

Here we prove the following:

Theorem 2. *If $f \in RO(\Omega)$, and φ is a conformal mapping from G onto Ω, then $f \circ \varphi \in G$. Moreover if $f \in OC^1_K(\Omega)$, then $f \circ \varphi$ is in $OC^1_C(G)$, where C depends only on K.*

Proofs

Our proofs are based on two theorems of Koebe (see [2, Theorem 2.3 and Theorem 2.5].

Theorem C (Koebe one-quarter theorem). *Let φ be a conformal mapping from the disk $B(z_0, R)$ into $\mathbb{C}$, then the image $\varphi(B(z_0, R))$ contains the disk $B(\varphi(z_0), \rho)$, where $\rho = R|\varphi'(z_0)|/4$.*

Theorem D (Koebe distortion theorem). *Let φ be a conformal mapping from the disk $B(z_0, R)$ into $\mathbb{C}$, then there holds the inequalities*

$$\frac{R^2(R - |z - z_0|)}{(R + |z - z_0|)^3} \leq \frac{|\varphi'(z)|}{|\varphi'(z_0)|} \leq \frac{R^2(R + |z - z_0|)}{(R - |z - z_0|)^3}, \qquad z \in B(z_0, R).$$

Consequently if $|z - z_0| < R/2$, then

$$\frac{|\varphi'(z)|}{|\varphi'(z_0)|} \geq \frac{4}{27}.$$

Proof of Theorem 1.

Let $u \in QNS_K(\Omega)$ and φ a conformal mapping from G onto Ω. We have to find a constant C such that

$$\int_{B(z_0,r)} u(\varphi(z))\, dm(z) \geq u(\varphi(z_0)) r^2/C, \tag{†}$$

whenever $r < \operatorname{dist}(z, \partial G)$. Let $w_0 = \varphi(z_0)$ and $\rho = r|\varphi'(z_0)|/4$, and let $\psi : \Omega \mapsto G$ denote the inverse of φ. Then

$$\begin{aligned} \int_{B(z_0,r)} u(\varphi(z))\, dm(z) &= \int_{\varphi(B(z_0,r))} u(w)|\psi'(w)|^2\, dm(w) \\ &\geq \int_{B(w_0,\rho/2)} u(w)|\psi'(w)|^2\, dm(w), \end{aligned} \tag{5}$$

where we have applied the one-quarter theorem. Now we apply the distortion theorem to the function ψ to get $|\psi'(w)| \geq (4/27)|\psi'(w_0)|$, for $|w - w_0| < \rho/2$. It follows that

$$\begin{aligned} \int_{B(z_0,r)} u(\varphi(z))\, dm(z) &\geq (4/27)^2 |\psi'(w_0)|^2 \int_{B(w_0,\rho/2)} u(w)\, dm(w) \\ &\geq (4/27)^2 |\psi'(w_0)|^2 (\rho/2)^2 u(w_0)/K \\ &= (4/27)^2 |\psi'(w_0)|^2 |\varphi'(z_0)|^2 u(w_0) r^2/16K. \end{aligned} \tag{6}$$

Now we use the identity $\psi'(w_0)\varphi'(z_0) = 1$ to get (†) with $C = 27^2K$. This concludes the proof of Theorem 1.

Proof of Theorem 2.

Let $u \in OC^1_K(\Omega)$ and φ a conformal mapping from G onto Ω. We have to find a constant C_1 such that

$$|\nabla u(\varphi(z_0))| \cdot |\varphi'(z_0)| \le \frac{C_1}{\varepsilon} \sup_{z\in B(z_0,\varepsilon)} |u(\varphi(z)) - u(\varphi(z_0))|, \qquad B(z_0,\varepsilon) \subset G. \tag{7}$$

Let $w_0 = \varphi(z_0)$ and $\rho = \varepsilon|\varphi'(z_0)|/4$, and let ψ be the inverse of φ. Then, the definition of OC^1_K and by the one-quarter theorem,

$$\begin{aligned}
&\sup\{|u(\varphi(z)) - u(\varphi(z_0))| : z \in B(z_0,\varepsilon)\}\\
&\quad = \sup\{|u(w) - u(w_0)| : w \in \varphi(B(z_0,\varepsilon))\}\\
&\quad \ge \sup\{|u(w) - u(w_0)| : w \in B(w_0,\rho)\}\\
&\quad \ge |\nabla u(w_0)|\rho/K\\
&\quad = |\nabla u(w_0)| \cdot |\varphi'(z_0)|\varepsilon/4K.
\end{aligned}$$

This gives (7) with $C_1 = 4K$, concluding the proof.

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