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ON GREEN FUNCTION FOR THE FREE PARTICLE

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Abstract

Application of the p-adic analysis and Green function in relativistic quantum theory is considered. Integrability of the free relativistic particle model is discussed. A convenient gauge is fixed. The Green function, as a corresponding propagator, are calculated in 4-dimensional p-adic Minkowski space. For this model, detailed calculation is performed for all three different cases: $p \equiv 1(\text{mod})4$, $p \equiv 3(\text{mod})4$ and $p = 2$.

1 Introduction

p-Adic numbers, introduced by Kurt Hansel at the end of 19^{th} century, and p-adic analysis, have a quite long history and reach bibliography devoted to investigation and application of this field of mathematics. During the last decades this, at least at the beginning pretty exotic, part of pure number theory has found its place in surprisingly many different fields of science and technology. Maybe the best guide to a reader interested in this topic would be the monography [1], even more than ten years old one.

In this paper we are mostly interested in application of p-adic mathematics in theoretical physics. More precisely, our goal is to calculate Green function for p-adic free relativistic particle (FRP), motivated by, so called, Hartle-Hawking (HH) approach to quantum cosmology. Namely, most of the p-adic physicists believe that the most important applications of p-adic analysis concerns the Planck scale physics and non-Archimedean structure at very small space-time distances.

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Foundations of p-adic quantum mechanics [2] and quantum cosmology [3] were worthwhile achievements in attempt of deeper understanding the fundamental processes at fundamental-Planck distance. The adelic connection between ordinary and p-adic quantum theory has been proposed for strings [4], quantum mechanics [5] and quantum cosmology [6].

Dynamics of some simple non-relativistic models is considered in a few papers: free particle and particle in constant external field [7], harmonic oscillator with constant [5] and time-dependent frequency [8]. Also, de Sitter and other interesting cosmological models have been considered [6].

In all that cases, functional integral approach and the corresponding Gauss integrals have been used [9]. As usually, while developing a theory, as (p-adic) quantum cosmology, it is very convenient to study some technically simpler cases. The FRP is formally used as one very instructive and integrable model. Although usual action for the relativistic particle is nonlinear, as in the real case [10], that system may be treated like a system with quadratic constraint. This paper is concerned about the p-adic aspects of such system. For a bit different and interesting approaches to p-adic Green function and p-adic free particle see [11].

The paper is organized as follows. After the Introduction we briefly recapitulate basic facts about p-adics. Section 3 is devoted to classical and quantum FRP over real numbers. In Section 4 p-adic propagator (Green function) for the FRP is introduced and studied in details. The main results are formulated in the form of three propositions.

2 p-Adic Mathematics

In considering p-adic numbers it is suitable to begin with the field of rational numbers Q, as the simplest field of numbers of characteristics 0. Any rational number can be expanded into one of two forms of infinite series [1]

$$\sum_{i}^{-\infty} a_i 10^i, a_i = 0, ..., 9, \tag{1}$$

or

$$\sum_{i=m}^{+\infty} b_i p^i, b_i = 0, ..., p-1, \tag{2}$$

where n and m are some integers, and p is a prime number. The above series (1) and (2) are convergent in respect to standard absolute value $|\ |_\infty$ and p-adic norm $|\ |_p$, respectively. From physical point of view, it is important that Q contains *all* results of physical measurements.

The completion of the field of rational numbers Q with respect to the standard norm $|\ |_\infty$ leads to the field of real numbers $R \equiv Q_\infty$ (expression (1)). According to the Ostrowski theorem, besides absolute value and p-adic norms $|\ |_p$ there are no other non-equivalent and nontrivial norms on Q. The completion of Q with respect to the p-adic norm leads to the p-adic number field Q_p (expression (2)). The p-adic norm is nonarchimedean (ultrametric) one, i.e. $|x+y|_p \leq \max(|x|_p, |y|_p)$.

Generally speaking, there are two analyses over Q_p, $\varphi : Q_p \to Q_p$ and $\psi : Q_p \to C$. In the case of mapping $\psi : Q_p \to C$, there is no standard derivative, and some types of pseudodifferential operators have been introduced [1, 12]. However, it turns out that there is a well defined integral with the Haar measure. In the following, we will use the integral formula

$$\int\limits_{S_\gamma} \chi_p(\xi y) dy = \begin{cases} p^\gamma \left(1 - p^{-1}\right), & \text{if} \quad |\xi|_p \leq p^{-\gamma} \\ -p^{\gamma-1}, & \text{if} \quad |\xi|_p = p^{-\gamma+1} \\ 0, & \text{if} \quad |\xi|_p \geq p^{-\gamma+2}. \end{cases} \tag{3}$$

where $\chi_p(u) = \exp(2\pi i u_p)$ is a p-adic additive character. Here, u_p denotes the fractional part of $u \in Q_p$. Recall that in the real case one has $\chi_\infty(x) = \exp(-2\pi i x)$. Also, let us state the notation for the ring of p-adic integers, p-adic circle and disc, respectively:

$$Z_p = \{x \in Q_p : |x|_p \leq 1\} \tag{4}$$

$$S_\gamma(a) = \{x \in Q_p : |x-a|_p = p^\gamma\}, \tag{5}$$

$$B_\gamma(a) = \{x \in Q_p : |x-a|_p \leq p^\gamma\}. \tag{6}$$

Even very interesting topics, adeles [13] and their application to physics [6, 9] will not be considered here.

3 Classical and Quantum Free Relativistic Particle

Usual action for the FRP, in the flat configuration space, is

$$S = -mc \int\limits_{\tau_1}^{\tau_2} d\tau \sqrt{\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}} \tag{7}$$

(a dot denotes a derivative of 4-vector x^μ with respect to τ , $\eta_{\mu\nu}$ is the usual Minkowski metric with signature $(-,+,+,+)$ and $\mu, \nu \in \{0,1,2,3\}$).

Speed of light in vacuum is denoted by c and m denotes particle mass. The quantity τ parameterizes the wordline, with the boundary condition

$$x(\tau_1) = x_1, \quad x(\tau_2) = x_2. \tag{8}$$

In using action (7) for the path integral quantization we meet with the problems [10]. First of all, there is the practical problem of evaluating a path integral which is not Gaussian.

However, FRP may be treated as a system with constraint [10] $H = \eta^{\mu\nu}k_\mu k_\nu + m^2c^2 = k^2 + m^2c^2 = 0$, ($k^\mu$ is the 4-momentum, $k^\mu = (E/c, \vec{k})$, E is a energy of particle and $\vec{k}$ is a 3-momentum). This Hamiltonian after introduction Lagrange multiplier N, is getting the canonical form

$$H_c = N(k^2 + m^2c^2). \tag{9}$$

The canonical Hamiltonian (9) leads to the Lagrangian

$$L = \dot{x}_\alpha k^\alpha - H_c, \tag{10}$$

where $\dot{x}_\alpha = \frac{\partial H_c}{\partial k^\alpha} = 2N\eta_{\mu\alpha}k^\mu$, $(k^\alpha = \frac{\dot{x}_\mu \eta^{\mu\alpha}}{2N},\ k_\alpha = \frac{\dot{x}_\alpha}{2N})$. For the action we gain

$$S = \int_{\tau_1}^{\tau_2} d\tau \left[\frac{\dot{x}^2}{4N} - m^2c^2N\right]. \tag{11}$$

The classical trajectory, as a solution of Euler-Lagrange equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$, with the boundary condition (8)

$$\bar{x} = \frac{x_2 - x_1}{\tau_2 - \tau_1}\tau + \frac{x_1\tau_2 - x_2\tau_1}{\tau_2 - \tau_1}, \tag{12}$$

leads to the classical action

$$\bar{S} = \frac{(x_2 - x_1)^2}{4N(\tau_2 - \tau_1)} - m^2c^2N(\tau_2 - \tau_1). \tag{13}$$

In quantization by the functional (Feynman) approach, first of all, we have to calculate the corresponding kernel of the operator of evolution U for the considered system. For the class of reparametrization-invariant systems, in the standard case, it can be presented as a functional part [10] of the

$$G(x_j|x_i) = \int dN(\tau_j - \tau_i) \int \mathcal{D}k\mathcal{D}x e^{i\int_{\tau_i}^{\tau_j} \tau[k\dot{x} - NH]}. \tag{14}$$

After the usual redefinition $T = N(\tau_j - \tau_i)$ we gain

$$G(x_j|x_i) = \int dT\mathcal{K}(x_j, T|x_i, 0) \tag{15}$$

where $G(x_j|x_i)$ is the propagator or Green function and $\mathcal{K}(x_j, T|x_i, 0)$ is above mentioned kernel.

Let us note that the Green function $G(x_j|x_i)$ for the $x_j = x$ and $x_i = 0$, in the quantum cosmology is the HH wave function of the universe (see [3]). The main results concerning the T integration in the standard case are presented in the Ref. 10.

4 p-Adic Propagator (Green function)

In the p-adic case, the above expressions from (7) to (15) are valid too. Since classical action (13) may be presented in the form

$$\bar{S} = -\frac{(x_2^0 - x_1^0)^2}{4T} - \frac{m^2c^2T}{4} + \frac{(x_2^1 - x_1^1)^2}{4T} - \frac{m^2c^2T}{4}$$

$$+\frac{(x_2^2 - x_1^2)^2}{4T} - \frac{m^2c^2T}{4} + \frac{(x_2^3 - x_1^3)^2}{4T} - \frac{m^2c^2T}{4} = \bar{S}^0 + \bar{S}^1 + \bar{S}^2 + \bar{S}^3 \tag{16}$$

corresponding p-adic kernel is [9]

$$\mathcal{K}_p(x_j, T_j|x_i, T_i) = \frac{\lambda_p^2\left(4h(T_j - T_i)\right)}{|2h(T_j - T_i)|_p^2}\chi_p\left(-\frac{1}{h}\frac{(x_j - x_i)^2}{4(T_j - T_i)} + \frac{m^2c^2}{h}(T_j - T_i)\right), \tag{17}$$

where $\lambda_p(u)$ is an arithmetic complex valued function [1].

Remind that $\mathcal{K}_p(x_j, T_j|x_i, T_i)$ is the kernel of the p-adic evolution operator which is defined by its action on the wave function $\psi(x_j)$

$$U_p(T)\psi_p(x_j) = \int_{Q_p} \mathcal{K}_p(x_j, T_j|x_i, T_i)\psi_p(x_i)dx_i. \tag{18}$$

Fixing the gauge $\dot{N} = 0$, we will calculate p-adic propagator as

$$G_p(x_j|x_i) = \int\limits_{|hT|_p \leq 1} dTK_p(x_j, T|x_i, 0), \tag{19}$$

i.e.,

$$G_p(x_j|x_i) = \int\limits_{|hT|_p\leq 1} dT \frac{\lambda_p^2(4hT)}{|2hT|_p^2}\chi_p\left(-\frac{(x_j-x_i)^2}{4hT}+\frac{m^2c^2}{h}T\right). \tag{20}$$

The integration is performed over the p-adic ball, $|hT|_p \leq 1$.

The nature of integration force us to divide integration in three different cases: 1) $p \equiv 1(mod)4$, $\lambda_p^2(a) = 1$, $p \equiv 3(mod)4$, $\lambda_p^2(a) = \pm 1$, 3) $p = 2$, $\lambda_p^2(a) = (-1)^{a_1} i$.

PROPOSITION 1. *For* $p \equiv 1(mod)4$, *i.e.* $\lambda_p^2(a) = 1$ *corresponding* p*-adic Green function reads*

$$G_p(x_j|x_i) = \begin{cases} -\frac{1}{p|h|_p}, & if \quad |(x_j-x_i)^2|_p \leq p \\ 0, & if \quad |(x_j-x_i)^2|_p > p. \end{cases} \tag{21}$$

To prove the above proposition, we introduce following changes

$$hT = z \Rightarrow dz = |h|_p dT; \quad z = \frac{1}{y} \Rightarrow dz = \frac{dy}{|y|_p^2} \Rightarrow dT = \frac{1}{|h|_p}\frac{dy}{|y|_p^2}. \tag{22}$$

Then, the Green function can be evaluated as (with $q^2 = \frac{(x_j-x_i)^2}{4}$)

$$G_p(x_j|x_i) = \frac{1}{|h|_p}\sum_{\gamma=0}^{\infty}\int\limits_{S_\gamma}\chi_p(-q^2y)dy. \tag{23}$$

Let $|q^2|_p \geq p^2$, than $|q^2|_p \geq p^{-\gamma+2}$, $\forall\gamma \geq 0$, $\gamma \in N+\{0\}$. In respect to the third line in (3), it follows

$$\sum_{\gamma=0}^{\infty}\int\limits_{S_\gamma}\chi_p(-q^2y)dy = 0 \Rightarrow G_p(x_j|x_i) = 0, \quad \text{for } |(x_j-x_i)^2|_p \geq p^2. \tag{24}$$

Let now $|q^2|_p = p^\delta$, $\delta \in Z\backslash N$

$$\sum_{\gamma=0}^{\infty}\int\limits_{S_\gamma}\chi_p(-q^2y)dy = \sum_{\gamma=0}^{-\delta}p^\gamma\left(1-\frac{1}{p}\right) - p^{|\delta|} = -\frac{1}{p}. \tag{25}$$

That means

$$G_p(x_j|x_i) = -\frac{1}{p}\frac{1}{|h|_p}, \quad for \quad |(x_j - x_i)^2|_p \leq p^\delta, \;\; \delta \in Z\backslash N. \tag{26}$$

On the light cone *e.g.* $q^2 = 0$

$$\sum_{\gamma=0}^{\infty}\int_{S_\gamma} 1dy = \sum_{\gamma=0}^{\infty} p^\gamma \left(1 - \frac{1}{p}\right) = \frac{1}{p}\left(p - 1 + p(p-1) + p^2(p-1)...\right) = -\frac{1}{p}. \tag{27}$$

All these results, in fact, are represented in the compact form (21).

PROPOSITION 2. *For* $p \equiv 3(mod)4$, *i.e*, $\lambda_p^2(a) = \pm 1$, *corresponding p-adic Green function reads*

$$G_p(x_j|x_i) = \frac{1}{|h|_p}\begin{cases} 0, & if \quad |(x_j - x_i)^2|_p \geq p^2, \\ -p^{-1}, & if \quad |(x_j - x_i)^2|_p = p, \\ (2 - p^{-1}), & if \quad |(x_j - x_i)^2|_p = 1, \\ \frac{1}{p+1}\left(\frac{p-1}{p} + (-1)^\delta 2p^{|\delta|+1}\right), & if \quad |(x_j - x_i)^2|_p = p^\delta, \delta < 0, \\ -\infty, & \text{if} \quad |(x_j - x_i)^2|_p = 0. \end{cases} \tag{28}$$

The most important difference in respect to $p \equiv 1(\text{mod})4$ is

$$\lambda_p^2(4hT) = \lambda_p^2(z) = \lambda_p^2(y) = (-1)^\gamma, \qquad |y|_p = p^\gamma. \tag{29}$$

Therefore, we calculate Green function as sum of two parts

$$G_p(x_j|x_i) = |h|_p^{-1}\left(\sum_{\gamma=0}^{+\infty}(-1)^{2\gamma}\int_{S_{2\gamma}} dy\chi_p\left(-q^2 y\right) + \sum_{\gamma=0}^{+\infty}(-1)^{2\gamma+1}\int_{S_{2\gamma+1}} dy\chi_p\left(-q^2 y\right)\right). \tag{30}$$

Let $|q^2|_p = p^\delta$, for $\delta \geq 2$. Follows

$$G_p(x_j|x_i) = 0 \quad for \quad |x_j - x_i|_p \geq p. \tag{31}$$

For $\delta = 1$, $G_p(x_j|x_i) = -\frac{1}{p|h|_p}$, and for $\delta = 0$, $G_p(x_j|x_i) = \frac{1}{|h|_p}(2 - \frac{1}{p})$.

For odd $\delta, \delta < 0$

$$G_p(x_j|x_i) = |h|_p^{-1}\sum_{\gamma=0}^{+\infty}(-1)^\gamma\int_{S_\gamma} dy\chi_p\left(-q^2 y\right)$$

$$= |h|_p^{-1}(p+1)^{-1}\left(\frac{p-1}{p} - 2p^{|\delta|+1}\right). \tag{32}$$

For even $\delta, \delta < 0$, one has

$$G_p(x_j|x_i) = |h|_p^{-1} \sum_{\gamma=0}^{-\delta} (-1)^\gamma p^\gamma (1 - p^{-1}) - p^{-\delta}(-1)^{-\delta+1} \tag{33}$$

$$= |h|_p^{-1}(p+1)^{-1} \left(\frac{p-1}{p} + 2p^{|\delta|+1} \right). \tag{34}$$

Finally, on the light cone we find

$$G_p(x_j|x_i) = -\frac{(p-1)^2}{p|h|_p^{-1}} \sum_{\gamma=0}^{+\infty} p^{2\gamma} = -\infty. \tag{35}$$

Collecting all above results we get form given in the PROPOSITION 2.

PROPOSITION 3. *In the case $p = 2$ (unique even prime number), i.e $\lambda_p^2(a) = (-1)^{a_1} i$, Green function for FRP has the following form*

$$G_2(x_j|x_i) = \begin{cases} 0, & if \quad q^2 = 0 \quad or \quad |(x_j - x_i)^2|_2 \geq 1, \\ \frac{(-1)^{q_1^2}}{|h(x_j - x_i)^2|_2}, & if \quad q^2 \neq 0 \quad and \quad |(x_j - x_i)^2|_2 \leq \frac{1}{2}. \end{cases} \tag{36}$$

To prove above proposition we start with

$$G_2(x_j|x_i) = |2h|_2^{-1} \int\limits_{|y|_2 \geq 2} dy \lambda_2^2(y) \chi_p\left(-q^2 y\right). \tag{37}$$

For $y = 2^{-\gamma}(1 + 2y_1 + ...), \gamma \in N, \quad \lambda_2^2(y) = (-1)^{y_1} i$, and

$$G_2(x_j|x_i) = |2h|_2^{-1} \sum_{\gamma=1}^{\infty} \left(i \int\limits_{S_{\gamma, y_1=0}} dy \chi_p(-q^2 y) - i \int\limits_{S_{\gamma, y_1=1}} dy \chi_p(-q^2 y) \right). \tag{38}$$

By the introducing I_0 and I_1 [1]

$$I_0 = \int\limits_{S_{\gamma, y_1=0}} dy \chi_p(-q^2 y), \quad and \quad I_1 = \int\limits_{S_{\gamma, y_1=1}} dy \chi_p(-q^2 y). \tag{39}$$

we finaly calculate
1)If $q^2 = 0, \quad I_0 = I_1 \quad \Rightarrow \quad G_2(x_j|x_i) = 0.$

2)If $|q^2|_2 = 2^\delta, \delta \in$ Z, and $|y|_2 \leq 2^{-\delta}$, $I_0 = I_1$.
3)If $|q^2|_2 = 2^\delta$, and $|y|_2 = 2^{-\delta+1}, \{q^2 y\}_2 = \frac{1}{2}$, $I_0 = I_1$.
4)If $|q^2|_2 = 2^\delta$, and $|y|_2 = \frac{1}{2^{\delta-2}}, \{q^2 y\}_2 = \frac{1}{4} + \frac{1}{2}(y_1 + q_1^2)$, $I_0 = \frac{-i(-1)^{q_1^2}}{2^\delta} = -I_1$.
5)If $|q^2|_2 = 2^\delta$, and $|y|_2 \geq 2^{-\delta+3}$, $I_0 = I_1$.
We have to keep in mind that the range of integration is $\gamma \geq 1$, so for unique nonsingular case 4) we have $-\delta + 2 \geq 1$, i.e. $\delta \leq 1$. It corresponds $|q^2|_2 = |\frac{(x_j - x_i)^2}{4}|_2 \leq 2 \Leftrightarrow |(x_j - x_i)^2|_2 \leq \frac{1}{2}$. Comparing 1) to 5) with (36) we finish the proof of the PROPOSITION 3.

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References

[1] V. S. Vladimirov, I. V. Volovich and E. I. Zelenov, *p-Adic Analysis and Mathematical Physics* (World Scientific, Singapore, 1994).

[2] V. S. Vladimirov and I. V. Volovich, *A vacuum state in p-adic quantum mechanics*, Phys. Lett. **B217**, 411 (1989).

[3] I. Ya. Aref'eva, B. Dragovich, P. Frampton and I. V. Volovich, *Wave function of the universe and p-adic gravity*, Int. J. Mod. Phys. **A6**, 4341 (1991).

[4] P. G. O. Freund and E. Witten, *Adelic string amplitudes*, Phys. Lett. **B199**, 191 (1987).

[5] B. Dragovich, *Adelic harmonic oscillator*, Int. J. Mod. Phys. **A10**, 2349 (1995).

[6] G. S. Djordjevic, B. Dragovich, Lj. Nesic, I.V.Volovich, *p-Adic and adelic minisuperspace quantum cosmology*, Int. J. Mod. Phys. **A17**, 1413 (2002).

[7] G. S. Djordjevic and B. Dragovich, *On p-adic functional integration*, Proc. of the II Mathematical Conference, Priština, Yugoslavia, 221 (1997); G. Djordjević, *On p-Adic and Adelic Quantum Mechanics*, University of Belgrade (1999) PhD Thesis (in Serbian).

[8] G. S. Djordjevic and B. Dragovich, *p-Adic and adelic harmonic oscillator with time-dependent frequency*, Theor. Mat. Phys. **124**, 1059 (2000).

[9] G. Djordjevic, B. Dragovich and Lj. Nesic, *Adelic path integrals for quadratic actions*, Infinite Dimensional Analysis, Quantum Probability and Related Topics, **6**, 179 (2003).

[10] J. J. Halliwell, *Derivation of the Wheeler-DeWitt equation from a path integral for minisuperspace models*, Phys. Rev. **D**, 2468 (1988).

[11] A. Kh. Bikulov, *Investigation of p-adic Green's function*, Theor. Math. Phys. **3** Vol. 87, 600 (1991); N. M. Chuong and N. Van Co, *The Multidimensional p-adic Green function*, Proc. of the American Math. Society **3** Vol. 127, 685 (1999); A. N. Kochubei, *On p-adic Green's functions*, Theor. Math. Phys. **1** Vol. 96, 854 (1991); A. Yu. Khrennikov, S. V. Kozyrev, *Localization in space for free particle in ultrametric quantum mechanics*, quant-ph/0508213.

[12] D. Dimitrijevic, G. S. Djordjevic and B. Dragovich, *On Schroedinger-type equation on p-adic spaces*, Bul. J. of Phys. **3** Vol. 27, 50 (2000).

[13] I. M. Gel'fand, M. I. Graev and I. I. Piatetskii-Shapiro, *Representation Theory and Automorphic Functions* (Saunders, London, 1966).

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