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ALMOST CONVERGENCE AND SOME MATRIX TRANSFORMATIONS

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Abstract

In this paper we characterize the matrix classes $(l(p,u), f_\infty)$ and $(l(p,u), f)$ which generalize the matrix classes given by Mursaleen [6].

1. Introduction

Let l_∞ and c be the Banach spaces of bounded and convergent sequences $x = (x_k)$ respectively with the usual norm $||x|| = \sup_k |x_k|$. A continuous linear functional ϕ on l_∞ is called a Banach limit [1] if (i) $\phi(x) \geq 0$ for $x = (x_k)$, $x_k \geq 0$ for every k, (ii) $\phi(x_{k+1}) = \phi(x_k)$, and (iii) $\phi(e) = 1$ where $e = (1, 1, 1, ...)$. A sequence $x \in l_\infty$ is said to be almost convergent to the value L if all of its Banach limits equal to L (see [3]). We denote the set of all almost convergent sequences by f, i.e.

$$f := \{x \in l_\infty : \lim_m t_{mn}(x) = L, \text{ uniformly in } n\},$$

where

$$t_{mn}(x) = \frac{1}{m+1} \sum_{k=0}^{m} x_{k+n} \quad , t_{-1,n} = 0,$$

and

$$L = f\text{-}\lim x.$$

Nanda [7] has defined a new set of sequences as follows

$$f_\infty := \{x \in l_\infty : \sup |t_{mn}(x)| < \infty\},$$

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we call f_∞ the set of all almost bounded sequences.

Bullet and Cakar [2] have defined a sequence space $l(p,s)$ and characterized the matrix classes $(l(p,s),l_\infty)$and $(l(p,s),c)$.

In this paper we characterize the matrix classes $(l(p,u),l_\infty)$ and $(l(p,u),f)$, where $l(p,u)$ is more general than $l(p,s)$.

Let $p=(p_k)$ be a sequence of real numbers with $p_k>0$. The space $l(p,s)$ is defined as (see [2])

$$l(p,s):=\{x:\sum_k k^{-s}|x_k|^{p_k}<\infty\}.$$

Let $u=(u_k)$ be a sequence of real numbers such that $u_k\neq 0$ ($k=1$, 2,....) and $u^{-1}=(u_k^{-1})$.

The space $l(p,u)$ is defined as (see [5])

$$l(p,u):=\{x:\sum_k |u_k x_k|^{p_k}<\infty\}.$$

If we take $u=(u_k)$ defined by

$$u_k=k^{-s/p_k} \quad , \quad s\geq 0 \quad , k=1,2,....$$

then $l(p,u)$ reduces to $l(p,s)$. Obviously $x\in l(p,u)$ has same meaning as $u.x\in l(p)$, and so $l(p,u)$ is paranormed by

$$g(x)=[\sum_k |u_k x_k|^{p_k}]^{1/M} \quad , \quad M=\max(1,\sup p_k).$$

Let X and Y be two sequence spaces and $A=(a_{nk})_{n,k=1}^\infty$ be an infinite matrix of real or complex numbers. We write $Ax=(A_n(x))$ if $A_n(x)=\sum_k a_{nk}x_k$ converges for each n. If $x=(x_k)\in X$ implies that $Ax\in Y$, then we say that A defines a matrix transformation from X into Y. By (X,Y) we denote the class of matrices A such that $Ax\in Y$ for $x\in X$.

2. Main Results

Throughout the text, we use the following notation :

For all integers $n, m \geq 1$,

$$t_{mn}(Ax) = \frac{1}{m+1} \sum_{i=0}^{m} A_{n+i}(x)$$

$$= \sum_{k} a(n,k,m) x_k$$

where

$$a(n,k,m) = \frac{1}{m+1} \sum_{i=0}^{m} a_{n+i,k}$$

Theorem 2.1. Let $1 < p_k < \sup_k p_k = H < \infty$ for every k. Then $A \in (l(p,u), f_\infty)$ if and only if there exists an integer $N > 1$ such that

(2.1.1) $$\sup_{m,n} \sum_k |a(n,k,m)|^{q_k} u_k^{-q_k} N^{-q_k} < \infty.$$

Proof. *Sufficiency.* Let (2.1.1) hold and that $x \in l(p,u)$ using the following inequality (see [4])

$$|ab| \leq C(|a|^q C^{-q} + |b|^p)$$

for $C > 0$ and a, b two complex numbers $(q^{-1} + p^{-1} = 1)$, we have

$$|t_{mn}(Ax)| = \sum_k |a(n,k,m) u_k^{-1} u_k x_k|$$

$$\leq \sum_k N[\ |a(n,k,m)|^{q_k} u_k^{-q_k} N^{-q_k} + |u_k x_k|^{p_k}],$$

where $q_k^{-1} + p_k^{-1} = 1$. Taking the supremum over m, n on both sides and using (2.1.1), we get $Ax \in f_\infty$ for $x \in l(p,u)$, i.e. $A \in (l(p,u), f_\infty)$.

Necessity. Let $A \in (l(p,u), f_\infty)$. Write $q_n(x) = \sup_m |t_{mn}(Ax)|$. It is easy to see that for $n \geq 0$, q_n is a continuous seminorm on $l(p,u)$ and (q_n) is pointwise bounded on $l(p,u)$. Suppose that (2.1.1) is not true. Then there exists $x \in l(p,u)$ with $\sup_n q_n(x) = \infty$. By the principle of condensation of singularities [8], the set

$$\{x \in l(p,u) : \sup_n q_n(x) = \infty\}$$

is of second category in $l(p,u)$ and hence nonempty, that is, there is $x \in l(p,u)$ with $\sup_n q_n(x) = \infty$. But this contradicts the fact that (q_n) is pointwise bounded on $l(p,u)$. Now by the Banach-Steinhaus theorem, there is constant M such that

$$q_n(x) \leq M g(x). \tag{2.1.2}$$

Now define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} \delta^{M/p_k} \; (sgn \;\; a(n,k,m)) \;\; |a(n,k,m)|^{q_k - 1} \; u_k^{-1} \; S^{-1} \; N^{-q_k/p_k}, & 1 \leq k \leq k_0 \\ 0 \quad \text{for} \quad k > k_0 \; ; \end{cases}$$

where $0 < \delta < 1$ and

$$S = \sum_{k=1}^{k_0} |a(n,k,m)|^{q_k} N^{-q_k}.$$

Then it is easy to see that $x \in l(p,u)$ and $g(x) \leq \delta$. Applying this sequence to (2.1.2) we get the condition (2.1.1).

Theorem 2.2. Let $1 < p_k \leq \sup_k p_k = H < \infty$ for every k. Then $A \in (l(p,u), f)$ if and only if

(i) The condition (2.1.1) of Theorem 2.1 holds;

(ii) $\lim_m a(n,k,m) = \alpha_k$ uniformly in n, for every k.

Proof. *Sufficiency.* Let (i) and (ii) hold and $x \in l(p,u)$.
For $j \geq 1$

$$\sum_{k=1}^{j} |a(n,k,m)|^{q_k} \; N^{-q_k} u_k^{-q_k}$$
$$\leq \sup_m \sum_k |a(n,k,m)|^{q_k} \; N^{-q_k} u_k^{-q_k} < \infty \text{ for every } n.$$

Therefore

$$\sum_k |\alpha_k|^{q_k}\ N^{-q_k}\ u_k^{-q_k} = \lim_j \lim_m \sum_{k=1}^{j} |a(n,k,m)|^{q_k}\ N^{-q_k}\ u_k^{q_k}$$

$$\leq \sup_m \sum_k |a(n,k,m)|^{q_k}\ N^{-q_k}\ u_k^{-q_k} < \infty, \quad (q_k^{-1} + p_k^{-1} = 1).$$

Consequently reasoning as in the proof of the sufficiency of Theorem 2.1, the series $\sum_k a(n,k,m)x_k$ and $\sum_k \alpha_k x_k$ converge for every n, m; and for every $x \in l(u,p)$. For a given $\epsilon > 0$ and $x \in l(u,p)$, choose k_0 such that

$$\left(\sum_{k=k_0+1}^{\infty} |u_k x_k|^{p_k}\right)^{\frac{1}{H}} < \epsilon, \tag{2.2.1}$$

where $H = \sup p_k$. Codition (ii) implies that there exists m_0 such that

$$\left|\sum_{k=1}^{k_0}[a(n,k,m) - \alpha_k]\right| < \epsilon/2 \ \text{ (for all } m \geq m_0), \text{ uniformly in } n. \tag{2.2.2}$$

Now, since $\sum_k a(n,k,m)x_k$ and $\sum_k \alpha_k x_k$ converges (absolutely) uniformly in m, n and for every $x \in l(u,p)$; we have that

$$\sum_{k=k_0+1}^{\infty} [a(n,k,m) - \alpha_k]x_k$$

converges uniformly in m, n for every $x \in l(u,p)$.

Hence by conditions (i) and (ii)

$$\left|\sum_{k=k_0+1}^{\infty} [a(n,k,m) - \alpha_k]\right| < \epsilon/2 \ \text{ (for all } m \geq m_0), \text{ uniformly in } n.$$

Therefore

$$\left|\sum_{k=k_0+1}^{\infty} [a(n,k,m) - \alpha_k]x_k\right| \to 0 \ \ (m \to \infty)$$

uniformly in n, i.e.

$$\lim_m \sum_k a(n,k,m)x_k = \sum_k \alpha_k x_k, \tag{2.2.3}$$

uniformly in n.
This completes the proof.

Necessity. Let $A \in (l(p,u), f)$. Since $f \subset f_\infty$ (see [6]), condition (i) follows by Theorem 2.1. Since $e_k = (0, 0, ...1(k\text{-}th\ place), 0, 0, ...) \in l(p,u)$, condition (ii) follows immediately by (2.2.3).

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