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A GENERAL FIXED POINT THEOREM FOR CONVERSE COMMUTING MULTIVALUED MAPPINGS IN SYMMETRIC SPACES

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Abstract

In this paper a general fixed point theorem for converse commuting multivalued mappings, which generalize Theorems 2.1 and 2.2 from [3], is proved.

1 Introduction

Since Jungck [1] introduced the concept of compatible mappings, which generalize the notion of weakly commuting mappings introduced by Sessa [5], many interesting results have been obtained by various authors. Singh and Mishra [6] introduced the notion of weakly compatible mappings of a hybrid pair g,F. In a recent paper, Sahu, Imdad and Kumar [4] proved a common fixed point theorem for weakly compatible mappings in symmetrizable topological spaces. All the concepts which generalize the notion of compatible mappings are considering the situation of fg and gf (or Fg and gF) from the conditions f and g (or F and g). Lü [2] presented the concept of the converse commuting mappings which discused the relation from the reverse, and gave some fixed point theorems for single valued mappings. Recently, Qi-kuan Liu and Xin-qi-Hu [3] introduced the new concept of converse commuting multivalued mappings and proved some fixed point theorems for converse commuting multivalued mappings.

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2 Preliminaries

Definition 2.1. A symmetric on a set X is a real valued function d on X x X such that
(i) $d(x,y) \geq 0, \forall x,y \in X$ and d(x,y)=0 if and only if x=y,
(ii) $d(x,y) = d(y,x), \forall x,y \in X$.
Let d be a symmetric on X and D be the metric on 2^X introduced by the symmetric d as follows:
$D(A,B) = inf\{d(a,b) : a \in A, b \in B\}$
for all $A, B \in 2^X$, and $d(x,A) = inf\{d(x,y) : y \in A\}$
for all $x \in X$ and $A \in 2^X$
Let f,g be single valued mappings from X into itself and $F : X \to 2^X$ be a multivalued mappings.
Definition 2.2. f and g are called converse commuting [2] if for all $x \in X$, fgx=gfx implies fx=gx.
Definition 2.3. $t \in X$ is said to be a commuting point of f and g [2] if fgt=gft.

In [3] the authors extend Definitions 2.2 and 2.3 for hybrid pairs of mappings.
Definition 2.4. The mappings g and F are said to be converse commuting if for all $x \in X$, gFx=Fgx implies $gx \in Fx$
Definition 2.5. t is said to be a commuting point of g and F if Fgt=gFt.

The following theorems are proved in [3].
Theorem 2.1. Let $g : X \to X$ and $F : X - 2^X$ be converse commuting multivalued mappings. Suppose that there exists a commuting point of g and F and
$d(gx, gy) \leq \Phi(max\{d(Fx, Fy), d(Fx, gy), d(gy, Fy)\}$ for each $x, y \in X$. If Φ is nondecreasing on R_+ and $\phi(t) < t, \forall t > 0$ then there exists a common fixed point of F and g.
Theorem 2.2. Four mappings $f : X \to X, g : X \to X, F : X \to 2^X$ and $G : X \to 2^X$ satisfy
$d(fx, gy) \leq \Phi(max\{D(Fx, Gy), d(fx, Fx), d(Gy, gy), d(fx, Gy), d(gy, Fx)\})$
for each $x, y \in X$. Suppose that there exists a commuting point of f and F and a commuting point of g and G. If (f,F) and (g,G) are converse commuting mappings and Φ is nondecreasing on R_+ and $0 < \phi(t) < t, \forall t > 0$, then there exists a common fixed point of f,g,F and G.

The purpose of this paper is to prove a general theorem which generalize Theorems 2.1 and 2.2 for hybrid functions satisfying an implicit relation.

3 Implicit relation

Let $\mathcal{F}_6$ be the set of all real functions $F(t_1,...,t_6) : R_+^6 \to R$ satisfying the following conditions:
(F_1) : F is nonincreasing in variables t_2, t_5, t_6,
(F_2) : $F(t,t,0,0,t,t) > 0, \forall t > 0$.
Example 3.1. $F(t_1,...,t_6) = t_1 + k\ max\{t_2,t_3,t_4,\frac{1}{2}(t_5+t_6)\}$, where $k \in [0,1)$.
(F_1) : Obviously.
(F_2) : $F(t,t,0,0,t,t) = t(1-k) > 0, \forall t > 0$.
Example 3.2. $F(t_1,...,t_6) = t_1^2 - at_1(t_2+t_3+t_4) - bt_5t_6$, where $a,b > 0$ and $a+b < 1$.
(F_1) : Obviously.
(F_2) : $F(t,t,0,0,t,t) = t^2(1-(a+b)) > 0, \forall t > 0$.
Example 3.3. $F(t_1,...t_6) = t_1^3 - at_2^2 - b\frac{t_3^2t_4+t_5^2t_6}{t_3+t_4+1}$, where $a,b > 0$ and $a+b < 1$.
(F_1) : Obviously.
(F_2) : $F(t,t,0,0,t,t) = t^3(1-(a+b)) > 0,\ \forall t > 0$.
Example 3.4. Let $\Phi : R_+ \to R_+$ be a function such that $0 < \phi(t) < t$, $\forall t > 0$, non-decreasing on R_+ and
$F(t_1,...,t_6) = t_1 - \phi(max\{t_2,t_3,...,t_6\})$.
(F_1) : Obviously.
(F_2) : $F(t,t,0,0,t,t) = t - \phi(t) > 0, \forall t > 0$.
Remark 3.1. There exist functions $F \in \mathcal{F}_6$ which are increasing in variable t_3, t_4.

4 Main result

Theorem 4.1. Assume that four mappings $f,g : X \to X$ and $f,g : X \to 2^X$ satisfy the inequality
(4.1) $\phi(d(fx,gy), D(Fx,Gx), d(fx,Fx), d(gy,Gy), d(fx,Gy), d(gy,Fx)) \leq 0$
for each $x,y \in X^2$, where $\phi \in \mathcal{F}_6$. If (f,F) and (g,G) are converse commuting multivalued mappings and f and F have a commuting point and g and G have a commuting point, then there exists a common fixed point of f,g F and G.
Proof. Let u be the commuting point of f and F and v be the commuting point of g and G. Since f and F are converse commuting we have fFu=Ffu and $fu \in Fu$, hence $d(fu,Fu) = 0$. It follows that $ffu \in fFu = Ffu$, hence $d(ffu,Fu) = 0$. Similarly, we have $gv \in Gv, d(gv,Gv) = 0$ and

$ggv \in gGv = Ggv$, hence $d(ggv, Ggv) = 0$.

Let us show that $fu = gv$. If not, since

$$D(Fu, Gv) \leq d(fu, gv), d(Fu, gv) \leq d(fu, gv), d(fu, Gv) \leq d(fu, gv),$$

by (4.1) and (F_1) we have successively
$\phi(d(fu, gv), D(Fu, Gv), d(fu, Fu), d(gv, Gv), d(fu, Gv), d(gv, Fu) \leq 0$
$\phi(fu, gv), d(fu, gv), 0, 0, d(fu, gv), d(gv, fu) \leq 0$
a contradiction of (F_2). Hence fu=gv. We claim that fu is a fixed point of f. Suppose that $fu \neq ffu$. Then $d(fu, ffu) = d(ffu, fu) = d(ffu, gv)$ and by (4.1) and (F_1) we have successively
$\phi(d(ffu, gv), D(ffu, gv), d(ffu, Ffu), d(gv, Gv), d(ffu, Gv), d(gv, Ffu) \leq 0$
$\phi(d(ffu, fu), d(ffu, fu), 0, 0, d(ffu, gv), d(gv, ffu) \leq 0$
$\phi(d(ffu, fu), d(ffu, fu), 0, 0, d(ffu, fu), d(ffu, fu) \leq 0$
a contradiction of (F_2). Therefore, fu=ffu. Similarly we have gv=ggv. Since fu=gv, we have fu=gv=ggv=gfu and fu is a fixed point of g.

On the other hand, $fu = gv \in Ggv = Gfu$ and $ffu \in fFu = Ffu$. hence fu is a common fixed point of f,g,F and G.

Theorem 4.2. Assume that four mappings $f_1, f_2, g_1, g_2 : X \to X$ satisfy the inequality
(4.2) $\phi(d(f_1x, g_1y), d(f_2x, g_2y), d(f_1x, f_2x), d(g_1y, g_2y), d(f_1x, g_2y), d(g_1y, f_2x)) \leq 0$
If (f, f_1) and (g, g_1) are converse commuting mappings and f and f_1 have a commuting point and g_1 and g_2 have a commuting point, where $\phi \in \mathcal{F}_6$ then there exists an unique common fixed point of f_1, f_2, g_1 and g_2.

Proof. By Theorem 4.1 there exists a common fixed point of f_1, f_2, g_1 and g_2. Let $v \neq u$ another common fixed point of f_1, f_2, g_1 and g_2. Then by (4.2) we have successively
$\phi(d(f_1u, g_1v), d(f_2u, g_2v), d(f_1u, f_2u), d(g_1v, g_2v), d(f_1u, g_2v), d(f_2u, g_1v)) \leq 0$
$\phi(d(u, v), d(u, v), 0, 0, d(u, v), d(u, v)) \leq 0$
a contradiction of (F_2).

Corollary 4.1. Theorem 2.2.

Proof. The proof it follows by Theorem 4.1 and Ex. 3.4.

Corollary 4.2. Theorem 2.1.

Proof. Since $max\{d(Fx, Fy), d(Fx, gy), d(gy, Fy)\} \leq$
$max\{d(Fx, Fy), d(fx, Fx), d(gy, Fy), d(fx, Fy), d(gy, Fx)\}$
and $\phi(t)$ is non decreasing the proof it follows by Theorem 4.1 and Corollary 4.1 for f=g and F=G.

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