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CENTERS IN INSERTED GRAPHS

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Abstract

In this paper we study some concepts involving distance in inserted graphs with an emphasize on centers in inserted graphs. More precisely we prove that for every non-trivial connected graph H there exists a graph G such that H is the center of G and the inserted graph of H is the center of the inserted graph of G. Graphs which are the periphery of some inserted graph are characterized.

1 Introduction

Suppose an official has to find a suitable place for an emergency facility (such as a fire station) in a given traffic network. It is naturally to locate it in such a way that the distance to the furthest vertex will be as short as possible, hence to build the fire station in the center of the corresponding graph. This is a reason for which centers in graphs have been studied in many papers.

We consider ordinary graphs (finite, undirected, with no loops or multiple edges). Let G be a graph with vertex set V_G and edge set E_G. Each member of $V_G \cup E_G$ will be called an element of G. A graph G is called a trivial graph if it has a vertex set with single vertex and a null edge set. If e is an edge of a graph G with end vertices x and y, then we denote the edge e, by $e = xy$. By $d_G(x, y)$ we mean the distance between the vertices x and y in the graph G.

It is known that for each graph H there is a graph G having the center H and containing at most four noncentral vertices [4]. The minimum number of noncentral vertices $A(H)$ among graphs having the center H was found by

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Buckley, Miller and Slater [5] in the case when H is a tree. They have also shown that for each graph H with $n \geq 9$ vertices and an integer $k \geq n+1$ there exists a k-regular graph G having the center H. So far little is known about centers of special graphs. Clearly the center of a tree consists of either a single vertex or a pair of adjacent vertices. All seven central subgraphs admissible in maximal outerplanar graphs were listed by Proskurowski [7]. The greatest subgraph contains six vertices. Laskar and Shier [6] studied centers in chordal graphs. A good survey on centers can be found in the book [4].

We introduce the notions of box graph $B(G)$ and inserted graph $I(G)$ of a non-trivial graph G in [3].

The connections in distance properties of a graph and its inserted graph are investigated in this paper. Relations between the eccentricity of an edge and the eccentricity of its end vertices are provided. We prove that for every non-trivial connected graph H there is a graph G such that H is the center of G and the inserted graph of H is the center of the inserted graph of G. If the inserted graph of a graph H has the radius at least three, then the similar result holds for the periphery instead of center.

In §2, we recall some definitions and results to be used in this paper.

In §3, we prove some basic results related with center and diameter of a graph.

In §4, we show that each inserted graph can be a center of some inserted graph and for a non-trivial connected graph H there exist connected graphs G_i such that $H = C(G_i)$, $I(C(G_i)) = C(I(G_i))$, where $i = r(I(G)) - r(G)$.

In §5, we shall study the existence of inserted graphs with a given periphery.

2 Preliminaries

Definition 2.1 *A graph can be constructed by inserting a new vertex on each edge of G and the resulting graph is called a box graph of G, denoted by $B(G)$. For an edge e of G, $\overline{e}$ denotes the vertex of $B(G)$ corresponding to the edge e.*

Definition 2.2 *Let I_G be the set of all inserted vertices in $B(G)$. A graph $I(G)$with vertex set I_G is called the inserted graph in which any two vertices are adjacent if they are joined by a path of length two in $B(G)$. Moreover if $V_G = \{v_1, v_2, ..., v_n\}$ and $E_G = \{e_1, e_2, ..., e_m\}$ then*

$$V_{B(G)} = \{v_1, v_2, ..., v_n, \overline{e}_1, \overline{e}_2, ..., \overline{e}_m\}.$$

Definition 2.3 *Let G be a connected graph, v a vertex and f an edge of G. Then the eccentricity $e_G(v)$ of the vertex v is the distance to a vertex furthest from v in G. A vertex is eccentric to the vertex v if their distance equals to $e_G(v)$. The eccentricity $e_G(f)$ of the edge f equals to the eccentricity $e_{I(G)}(f)$ of the vertex $\overline{f}$ in the graph $I(G)$. The radius $r(G)$ is the minimal eccentricity of the vertices, whereas the diameter $d(G)$ is their maximal eccentricity.*

Definition 2.4 *The vertex v is said to be a central vertex of G if $e_G(v) = r(G)$. The center $C(G)$ of G is the subgraph induced by all central vertices, while the periphery $Per(G)$ of G is the subgraph induced by the vertices with the greatest eccentricity.*

The next theorem is due to Adhikari and Pramanik [2].

Theorem 2.5 *Let $e = ab$ and $f = uv$ be two different edges in a connected graph G. Then for their distance in the inserted graph of G we have*

$$d_{I(G)}(\overline{e}, \overline{f}) = 1 + min\{d_G(a, u), d_G(a, v), d_G(b, u), d_G(b, v)\}.$$

Let k be an integer and let x, y be two vertices of a graph G, we mean by $S_k(x, y)$ the subgraph in G induced by the vertices which have the distance from both x and y at least k. Now we can express the eccentricity in an inserted graph in the following way:

Observation 2.6 *Let u and v be adjacent vertices in a connected graph G with at least three vertices. Then the eccentricity of the vertex $\overline{uv}$ in $I(G)$ equals to the maximal $k \geq 0$ such that the subgraph $S_{k-1}(u, v)$ contains an edge.*

3 Basic results

We start this section with a lemma. The lemma provides relations between the eccentricity of an edge and that of its end vertices.

Lemma 3.1 *Let u and v be adjacent vertices of a connected graph G. Then $\mid e_{I(G)}(\overline{uv}) - e_G(v) \mid \leq 1$ holds. Moreover, if u and v have distinct eccentricities, then $\mid e_G(u) - e_G(v) \mid = 1$ holds and the eccentricity of the edge uv equals to the eccentricity of one of its end vertices.*

Proof: Note that $\mid e_G(u) - e_G(v) \mid \leq 1$ as u and v are adjacent. Now it is sufficient to prove that $\mid e_{I(G)}(\overline{uv}) - e_G(v) \mid \leq 1$ holds. If G has two vertices, then G is a complete graph with 2 vertices, i.e., K_2 and the lemma holds. Now we assume that G has at least two edges. Then there exists an edge distinct from uv and by Theorem 2.5 we have $e_{I(G)}(\overline{uv}) = e_G(uv) \leq 1 + e_G(v)$. Further, we verify that $e_{I(G)}(\overline{uv}) = e_G(uv) \geq e_G(v) - 1$ holds. Let a be a vertex eccentric to v and distinct from u and let b be a neighbour of a. Then the distance between any vertex from $\{u, v\}$ and any vertex from $\{a, b\}$ is at least $e_G(v) - 2$, since otherwise there will be a $v - a$ path with the length shorter than $e_G(v)$. As $uv \neq ab$, according to Theorem 2.5 we have $e_{I(G)}(\overline{uv}) = e_G(uv) \geq d_{I(G)}(\overline{uv}, \overline{ab}) \geq 1 + e_G(v) - 2 = e_G(v) - 1$.

Corollary 3.2 *For a connected graph G with at least three vertices we have $\mid r(I(G)) - r(G) \mid \leq 1$. Moreover, $r(I(G)) = r(G) + 1$, if and only if for each two adjacent central vertices x and y there is an edge f such that both end vertices of f are eccentric to both x and y. Further, $r(I(G)) = r(G) - 1$ if and only if for each edge f joining central vertices and each other edge g at least one end vertex of f has the distance at most $r(G) - 2$ to some end vertex of the edge g.*

Corollary 3.3 *For a connected graph G with at least three vertices we have $\mid d(I(G)) - d(G) \mid \leq 1$.*

A connected graph G is called self-centered, if $C(G) = G$. Now some consequences for the radius and the center in an inserted graph follow:

Theorem 3.4 *Let G be a connected graph with at least three vertices. If G has a nontrivial center and a radius greater than its inserted graph, then $C(I(G))$ is an induced subgraph of $I(C(G))$. Moreover, if $I(G)$ is self-centered then G is also self-centered.*

Proof: Let $r(I(G)) = R - 1$, where R is the radius of G. Then for a vertex $\overline{uv}$ in $C(I(G))$ we have $e_G(u) \geq R$ and $e_G(v) \geq R$, Lemma 3.1 gives $e_G(u) = e_G(u) = R$, hence $\overline{uv}$ is in $I(C(G))$. Moreover, if $I(G)$ is self-centered then $C(I(G)) = I(G)$ is an induced subgraph in $I(C(G))$, so G is a subgraph of $C(G)$ and G is self-centered.

Theorem 3.5 *Let G be a connected graph with at least three vertices. If G has a nontrivial center and a radius smaller than its inserted graph, then $I(C(G))$ is an induced subgraph of $C(I(G))$. Moreover, if G is self-centered then $I(G)$ is also self-centered, and $I(C(G)) = C(I(G))$ if and only if G is self-centered.*

Proof: Let $r(I(G)) = R + 1$, where R is the radius of G. Then for a vertex $\overline{uv}$ in $I(C(G))$ we have $e_G(u) = e_G(v) = R$, which gives $e_G(uv) =$

$e_{I(G)}(\overline{uv}) \leq R+1 = r(I(G))$, and that is why $\overline{uv}$ is in $C(I(G))$. Moreover, if G is self-centered then $I(G)$ is an induced subgraph in $C(I(G))$, hence $I(G)$ is self-centered. Further, if G is self-centered, then we have $I(C(G)) = I(G) = C(I(G))$.

On the other hand, suppose that G is not self-centered. Then it contains an edge cy joining a central vertex c to a noncentral vertex y, hence $e_G(c) = R$ and $e_G(y) = R+1$. Then $e_G(cy) = e_{I(G)}(\overline{cy}) \leq R+1$ due to Lemma 3.1, and $\overline{cy}$ is in $I(C(G))$. Hence $I(C(G)) = C(I(G))$ does not hold.

Theorem 3.6 *Let G be a connected graph with at least three vertices. If G has a nontrivial periphery and a diameter greater than its inserted graph, then $I(Per(G))$ is an induced subgraph of $Per(I(G))$. Moreover, if G is self-centered then $I(G)$ is also self-centered and $I(Per(G)) = Per(I(G))$ if and only if G is self-centered.*

Proof: Let $d(I(G)) = D-1$, where D is the diameter of G. Then for a vertex $\overline{uv}$ in $I(Per(G))$ we have $e_G(u) = e_G(v) = D$, so $e_G(uv) = e_{I(G)}(\overline{uv}) \geq D-1 = d(I(G))$, and which gives that $\overline{uv}$ is in $Per(I(G))$. Moreover, if G is self-centered then $Per(G) = G$ and $I(Per(G)) = I(G)$ is an induced subgraph in $Per(I(G))$, hence $I(G) = Per(I(G))$, which means that $I(G)$ is self-centered. Further, if G is self-centered, then $I(G)$ is also self-centered, and clearly $I(Per(G)) = Per(I(G))$ holds.

On the other hand, if G is not self-centered, then it contains an edge cy such that $e_G(c) = D$ and $e_G(y) = D-1$, clearly $\overline{cy}$ is not in $I(Per(G))$. Then, due to Lemma 3.1, we have $e_G(cy) = e_{I(G)}(\overline{cy}) \geq D-1 = d(I(G))$, hence $\overline{cy}$ is in $Per(I(G))$. Hence $I(Per(G)) = Per(I(G))$ does not hold.

Theorem 3.7 *Let G be a connected graph with at least three vertices. If G has a nontrivial periphery and a diameter smaller than its inserted graph, then $Per(I(G))$ is an induced subgraph of $I(Per(G))$. Moreover, if $I(G)$ is self-centered then G is also self-centered.*

Proof: Let $d(I(G)) = D+1$, where D is the diameter of G. Then for a vertex $\overline{uv}$ in $Per(I(G))$ we have $e_{I(G)}(\overline{uv}) = D+1 = e_G(uv)$, so $e_G(u) \geq D$ and $e_G(v) \geq D$, and u and v are in $Per(G)$. Hence $\overline{uv}$ is in $I(Per(G)$. Moreover, if $I(G)$ is self-centered then $I(G) = Per(G)$ and so G is a subgraph of $Per(G)$. Thus G is self-centered.

4 Inserted graphs with a prescribed center

In this section at first we show that each inserted graph can be a center of some inserted graph.

Theorem 4.1 *Let H be a connected graph with n vertices and $m \geq 1$ edges. Then there is a connected graph G with at most $4n$ vertices and at most $m + n(n+1)$ edges such that $I(H)$ is the center of $I(G)$.*

Proof: Let $V_H = \{v_1, v_2, ..., v_n\}$ be the vertex set of H. Now we will construct its supergraph G as follows. Its vertex set will be the set $\{v_i, x_i, y_i, z_i \mid i = 1, 2, ..., n\}$. Further the edge set consists of the edges of H together with the edges joining x_i to each vertex from $V_H - \{v_i\}$, the edges $x_i y_i$ and $y_i z_i$ for all $i = 1, 2, ..., n$ (see Fig.1 for $H = P_2$, where P_2 is a path of length 2).

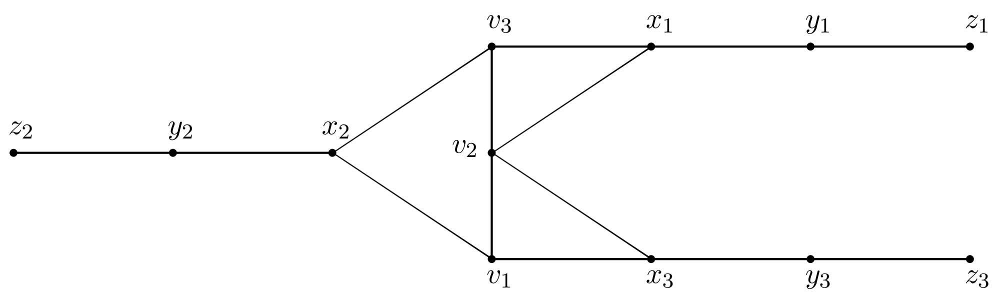

Figure 1 : The inserted graph of the drawn graph has $I(P_2)$ as its center.

Clearly, each edge joining two central vertices has the eccentricity three, while for any other edge f let say v_1 be a central vertex which is nearest to f. Then its distance to $y_1 z_1$ is at least four. Hence $C(I(G)) = I(H)$ holds.

Remark 4.2 *Under the condition of Theorem 4.1 there exist connected graphs with number of vertices less than $4n$ and number of edges less than $m + n(n+1)$ such that $I(H)$ is the center of $I(G)$. As an example we draw the Figure 2 in support of the remark.*

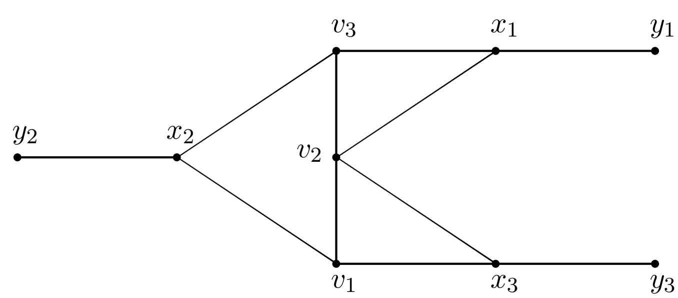

Figure 2 : The inserted graph of the drawn graph has $I(P_2)$ as its center.

Now we shall study connected graphs with a nontrivial center for which the mappings I and C commute, hence $I(C(G)) = C(I(G))$. Denote $\triangle r(G) =$

$r(I(G))-r(G)$. If $\triangle r(G)=1$ then due to Theorem 3.5, the mappings I and C commute if and only if G is self-centered. Complete graphs are examples of such graphs. But for any $i \in \{0,-1\}$ any graph H without isolated vertices, there is a graph G with $I(C(G)) = C(I(G))$ and $\triangle r(G) = i$, as the next theorem states.

Theorem 4.3 *Let H be a connected graph with n vertices, $m \geq 1$ edges and let i be either 0 or -1. Then there exist connected graphs G_i such that $H = C(G_i)$, $I(C(G_i)) = C(I(G_i))$ and $i = \triangle r(G)$ holds. Moreover, G_0 has $4n+6$ vertices and $m+n^2+4n+4$ edges.*

Proof: Let $V_H = \{v_1, v_2, ..., v_n\}$ be the vertex set of H and let E_H be its edge set. We shall construct graphs G_0 and G_{-1} with the requested properties and start with G_0.

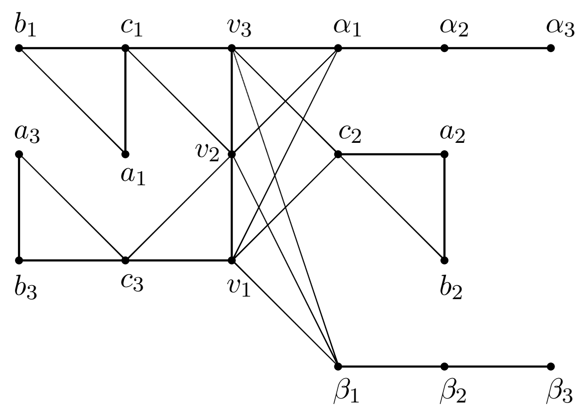

Figure 3: The graph G_0 with $C(G_0) = P_2$ and $C(I(G_0)) = I(C(G_0))$

The vertex set of G_0 equals $V_H \cup \{a_i, b_i, c_i \mid i = 1, 2, ..., n\} \cup \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\}$. Its edge set is $E_H \cup \{a_ib_i, a_ic_i, b_ic_i, \alpha_1 v_i, \beta_1 v_i \mid i = 1, 2, ..., n\} \cup \{c_i v_j \mid i \neq j\} \cup \{\alpha_1\alpha_2, \alpha_2\alpha_3, \beta_1\beta_2, \beta_2\beta_3\}$ (see Fig.3). One can check that $r(G_0) = 3$ and $H = C(G_0)$, since every vertex outside H has the distance at least four to some of the vertices α_3 and β_3. Further, any edge from $C(G_0)$ has the eccentricity three (its distance to a_1b_1 is three). As a center of $I(G_0)$ lies in a single block of $I(G_0)$, if any edge not from H lies in $C(I(G_0))$, then one its end vertex, say v_1, is in H. Then its distance to a_1b_1 is four, hence $C(I(G_0)) = I(C(G_0))$ holds.

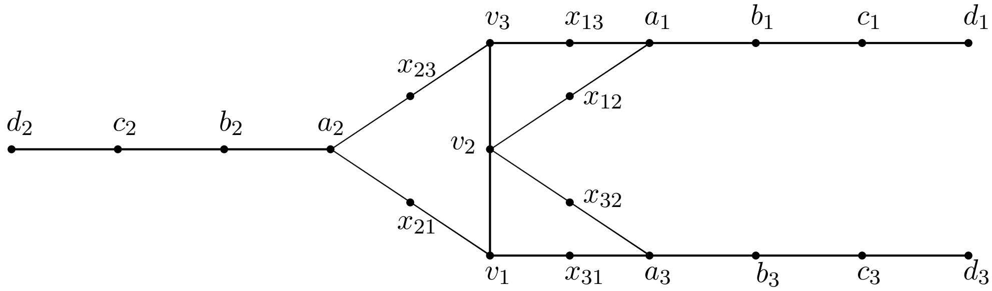

Figure 4: The graph G_{-1} with $C(G_{-1}) = P_2$ and $I(C(G_{-1})) = C(I(G_{-1}))$

Now we shall construct the graph G_{-1}. The vertex set of G_{-1} will contain $V_H \cup \{a_i, b_i, c_i, d_i \mid i = 1, 2, ..., n\} \cup \{x_{ij} \mid i \neq j, i = 1, 2, ..., n, j = 1, 2, ..., n\}$. Its edge set consists of $E_H \cup \{x_{ij}a_i, x_{ij}b_j \mid i \neq j\} \cup \{a_ib_i, b_ic_i, c_id_i \mid i = 1, 2, ..., n\}$ (see Fig.4).

Obviously, H is the center of G_{-1} and $r(G_{-1}) = 6$, as $d(v_i, d_i) = 6$. Note that every joining central vertices has the eccentricity five due to Theorem 2.5. Further, if an edge is adjacent to exactly one central vertex, say v_1, then its distance to the edge c_1d_1 is at least six. Finally, if an edge f is adjacent to no central vertex, then its distance to some edge of the form c_id_i is also at least six, so $r(I(G_{-1})) = 5 = r(G_{-1}) + 1$ and $I(H) = C(I(G_{-1}))$holds.

Theorem 4.4 *Let H be a connected bipartite graph of n vertices and $m \geq 1$ edges. Then there is a graph G with $n+6$ vertices and $m+n+4$ edges having the center H and satisfying $C(I(G)) = I(C(G))$.*

Proof: Let A and B be disjoint sets of vertices in H, such that adjacent vertices lie in distinct sets. We obtain G after the addition of the new vertices a, a_1, a_2, b, b_1 and b_2 such that a is adjacent to a_1 and to all vertices in A, b is adjacent to b_1 and to all vertices in B and a_2a_1 and b_1b_2 are also adjacent (see Fig.5). Clearly, G has the desired property.

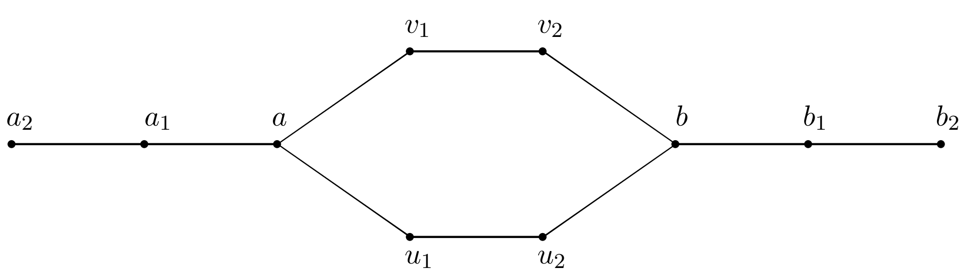

Figure 5 : A graph having the center $H = 2K_2$

5 The periphery in inserted graphs

Now we shall study the existence of inserted graphs with a given periphery. Note that $r(Per(G)) \geq d(G)$ holds for each graph G.

Theorem 5.1 *Let H be a nontrivial graph such that $I(H)$ has a radius at most two. Then $I(H)$ is the periphery of some inserted graph if and only if either $I(H)$ is self-centered or H contains two vertices which are not end vertices and each edge is adjacent to just one of them.*

Proof: If $I(H)$ is self-centered, then $I(H) = Per(I(H))$ holds. Now assume H contains two vertices u and v which are not end vertices and such that each edge is adjacent to just one of the vertices u and v. Then $I(H) = Per(I(H+uv))$ as if we add the edge uv to H, then its eccentricity will be one, while each other edge has the eccentricity two, since we have $d_{H+uv}(ux, vy)$ for pairwise distinct vertices x, y, u and v.

Assume now that there exists a graph G such that $I(H)$ is the periphery of $I(G)$. Then we have $2 \geq r(I(H)) \geq r(Per(I(G))) \geq d(I(G))$. Hence $I(G)$ is either self-centered or has the diameter two and the radius one. If $I(G)$ is self-centered, then $I(H) = Per(I(G)) = I(G)$, hence $I(H)$ has to be self-centered. Assume now the latter case holds. Then G contains an edge uv with the eccentricity one and so each edge is adjacent to u or v. Note that $I(H)$ has the radius two as we have $r(I(H)) \geq d(I(G)) = 2$. Hence the edge uv is not in H. Further, if u is an end vertex in H and x is the only its neighbour, then as H is connected there is a vertex $y, y \neq u$ adjacent to xv. But $y = v$ as the edge xy is adjacent to either u or v and x distinct from u and v. Hence xv has the eccentricity one which contradicts $r(I(H)) = 2$. So u and similarly v are not end vertices, which completes the proof.

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