Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **21:2** (2007), 31–44

# SOME GENERALIZATONS OF ALMOST CONTRA-SUPER-CONTINUITY

**Erdal Ekici**

**Abstract**

We introduced and studied some new classes of functions called $(e^*, s)$-continuous functions, $(e, s)$-continuous functions and $(a, s)$-continuous functions. These new notions of functions generalize the class of almost contra-super-continuous functions. Some properties and several characterizations of these types of functions are obtained. We investigate the relationships between these classes of functions and other classes of non-continuous functions.

## 1 Introduction

Recently, Ekici [10-12] has introduced new classes of sets called $e$-open sets, $e^*$-open sets and $a$-open sets to establish some new decompositions of continuous functions. By using new notions of $e$-continuous functions, $e^*$-continuous functions and $a$-continuous functions via $e$-open sets, $e^*$-open sets and $a$-open sets, respectively, Ekici has obtained some new decompositions of continuous functions. In this paper, we introduce new classes of functions called $(e^*, s)$-continuous functions, $(e, s)$-continuous functions and $(a, s)$-continuous functions which are generalizations of almost contra-super-continuous functions. We obtain some characteizations and several properties of such functions.

In this paper, spaces $X$ and $Y$ mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset $P$ of a space $X$, $cl(P)$ and $int(P)$ represent the closure of $P$ and the interior of

2000 *Mathematics Subject Classification.* 54C08.
*Key words and phrases.e-open set, $e^*$-open set, a-open set, $(e^*, s)$-continuous function, $(e, s)$-continuous function, $(a, s)$-continuous function.*
Received: November 8, 2006

$P$, respectively. A subset $P$ of a space $X$ is said to be regular open (resp. regular closed) if $P = int(cl(P))$ (resp. $P = cl(int(P))$) [24]. The $\delta$-interior [26] of a subset $P$ of $X$ is the union of all regular open sets of $X$ contained in $P$ and it is denoted by $\delta$-$int(P)$. A subset $P$ is called $\delta$-open if $P = \delta$-$int(P)$. The complement of $\delta$-open set is called $\delta$-closed. The $\delta$-closure of a set $P$ in a space $(X,\tau)$ is defined by $\delta$-$cl(P) = \{x \in X : P \cap int(cl(U)) \neq \emptyset$, $U \in \tau$ and $x \in U\}$ and it is denoted by $\delta$-$cl(P)$. A subset $P$ is said to be semi-open [15] ($\alpha$-open [18]) if $P \subset cl(int(P))$ ($P \subset int(cl(int(P)))$). The complement of a semi-open (resp. $\alpha$-open) set is called semi-closed [5] (resp. $\alpha$-closed [18]). The intersection of all semi-closed sets containing $P$ is called the semi-closure of $P$ and is denoted by $s$-$cl(P)$. A point $x \in X$ is said to be a $\theta$-semi-cluster point [14] of a subset $P$ of $X$ if $cl(U) \cap P \neq \emptyset$ for every semi-open set $U$ containing $x$. The set of all $\theta$-semi-cluster points of $P$ is called the $\theta$-semi-clusure of $P$ and is denoted by $\theta$-$s$-$cl(P)$. A subset $P$ is called $\theta$-semi-closed if $P = \theta$-$s$-$cl(P)$. The complement of a $\theta$-semi-closed set is called $\theta$-semi-open.

A subset $P$ of a space $X$ is said to be preopen [17] (resp. $\beta$-open [1], $\delta$-preopen [22], $\delta$-semiopen [21]) if $P \subset int(cl(P))$ (resp. $P \subset cl(int(cl(P)))$, $P \subset int(\delta\text{-}cl(P))$, $P \subset cl(\delta\text{-}int(P))$). The complement of a preopen (resp. $\beta$-open, $\delta$-semiopen, $\delta$-preopen) set is called preclosed (resp. $\beta$-closed, $\delta$-semiclosed, $\delta$-preclosed). The intersection of all preclosed (resp. $\alpha$-closed) sets, each containing a set $S$ in a topological space $X$ is called the preclosure (resp. $\alpha$-closure) of $S$ and it is denoted by $p$-$cl(S)$ ($\alpha$-$cl(S)$). For a subset $R$ of a space $X$, the set $\cap\{P \in RO(X) : R \subset P\}$ is called [6] the r-kernel of $R$ and is denoted by r-$ker(R)$.

**Lemma 1** *([6]) The following properties hold for $P \subset X$ and $R \subset X$:*

*(1) $x \in$r-$ker(P)$ if and only if $P \cap S \neq \varnothing$ for any regular closed set $S$ containing $x$.*

*(2) $P \subset$r-$ker(P)$ and $P =$r-$ker(P)$ if $P$ is regular open in $X$.*

*(3) $P \subset R$, then r-$ker(P) \subset$r-$ker(R)$.*

**Definition 2** *A subset $S$ of a space $(X,\tau)$ is called*

*(1) e-open [10] if $S \subset cl(\delta\text{-}int(S)) \cup int(\delta\text{-}cl(S))$ and e-closed [10] if $cl(\delta\text{-}int(S)) \cap int(\delta\text{-}cl(S)) \subset S$,*

*(2) $e^*$-open [11] if $S \subset cl(int(\delta\text{-}cl(S)))$ and $e^*$-closed [11] if $int(cl(\delta\text{-}int(S))) \subset S$,*

*(3) a-open [12] if $S \subset int(cl(\delta\text{-}int(S)))$ and a-closed [12] if $cl(int(\delta\text{-}cl(S))) \subset S$.*

The family of all $\delta$-open (resp. $e^*$-open, $e^*$-closed, regular open, regular closed, semi-open, closed, $e$-open, $a$-open) sets of $X$ containing a point $x \in X$ is denoted by $\delta O(X,x)$ (resp. $e^*O(X,x)$, $e^*C(X,x)$, $RO(X,x)$, $RC(X,x)$, $SO(X,x)$, $C(X,x)$, $eO(X,x)$, $aO(X,x)$). The family of all $\delta$-open (resp. $e^*$-open, $e^*$-closed, regular open, regular closed, semi-open, $\beta$-open, preopen, $e$-open, $a$-open) sets of $X$ is denoted by $\delta O(X)$ (resp. $e^*O(X)$, $e^*C(X)$, $RO(X)$, $RC(X)$, $SO(X)$, $\beta O(X)$, $PO(X)$, $eO(X)$, $aO(X)$).

The union of any family of $e^*$-open (resp. $e$-open) sets is an $e^*$-open (resp. $e$-open) set. The intersection of any family of $e^*$-closed (resp. $e$-closed) sets is an $e^*$-closed (resp. $e$-closed) set [10, 11]. The intersection of all $e^*$-closed (resp. $e$-closed, $a$-closed) sets containing $P$ is called the $e^*$-closure [11] (resp. $e$-closure [10], $a$-closure [12]) of $P$ and is denoted by $e^*\text{-}cl(P)$ (resp. $e\text{-}cl(P)$, $a\text{-}cl(P)$). The $e^*$-interior [11] (resp. $e$-interior [10], $a$-interior [12]) of $P$, denoted by $e^*\text{-}int(P)$ (resp. $e\text{-}int(P)$, $a\text{-}int(P)$), is defined by the union of all $e^*$-open (resp. $e$-open, $a$-open) sets contained in $P$.

**Lemma 3** *([11, 13]) The following hold for a subset $P$ of a space $X$:*

*(1) $e^*\text{-}cl(P)$ (resp. $e\text{-}cl(P)$, $a\text{-}cl(P)$) is $e^*$-closed (resp. $e$-closed, $a$-closed).*

*(2) $X \backslash e^*\text{-}cl(P) = e^*\text{-}int(X \backslash P)$ and $X \backslash e\text{-}cl(P) = e\text{-}int(X \backslash P)$ and $X \backslash a\text{-}cl(P) = a\text{-}int(X \backslash P)$.*

**Lemma 4** *([16]) $s\text{-}cl(N) = int(cl(N))$ for an open subset $N$ of a space $X$.*

## 2 Weak forms of almost contra-super-continuity

**Definition 5** *A function $f : X \to Y$ is called $(e^*, s)$-continuous (resp. $(e, s)$-continuous, $(a, s)$-continuous) if the inverse image of each regular open set of $Y$ is $e^*$-closed (resp. $e$-closed, $a$-closed) in $X$.*

**Definition 6** *A function $f : X \to Y$ is said to be*

*(1) contra R-map [9] if $f^{-1}(N)$ is regular closed in $X$ for every regular open set $N$ of $Y$,*

*(2) almost contra-super-continuous [7] if $f^{-1}(N)$ is $\delta$-closed in $X$ for every regular open set $N$ of $Y$,*

*(3) $(\delta\text{-}semi, s)$-continuous [6] if the inverse image of each regular open set of $Y$ is $\delta$-semiclosed in $X$,*

*(4) $(\delta\text{-}pre, s)$-continuous [8] if the inverse image of each regular open set of $Y$ is $\delta$-preclosed in $X$.*

**Remark 7** *The following diagram holds for a function $f : X \to Y$:*

$$
\begin{array}{ccccc}
 & & (e^*, s)\text{-}continuous & & \\
 & & \uparrow & & \\
 & & (e, s)\text{-}continuous & & \\
 & \nearrow & & \nwarrow & \\
(\delta\text{-}semi, s)\text{-}continuous & & & & (\delta\text{-}pre, s)\text{-}continuous \\
 & \nwarrow & & \nearrow & \\
 & & (a, s)\text{-}continuous & & \\
 & & \uparrow & & \\
 & & almost\ contra\text{-}super\text{-}continuous & & \\
 & & \uparrow & & \\
 & & contra\ R\text{-}map & &
\end{array}
$$

None of these implications is reversible as shown in the following examples.

**Example 8** *Let $X = \{a, b, c, d\} = Y$ and $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $\sigma = \{\varnothing, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(a) = d$, $f(b) = d$, $f(c) = d$, $f(d) = c$. Then, $f$ is $(a, s)$-continuous but it is not almost contra-super-continuous.*

**Example 9** *Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. Let $f : X \to X$ be a function defined by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then, $f$ is $(e, s)$-continuous but it is not $(\delta$-semi$, s)$-continuous.*

**Example 10** *Let $X = \{a, b, c, d\}$ and $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the identitiy function $i : X \to X$ is $(\delta$-semi$, s)$-continuous but it is not $(a, s)$-continuous. The function $f : X \to X$ defined by $f(a) = a$, $f(b) = c$, $f(c) = a$, $f(d) = c$ is $(e^*, s)$-continuous but it is not $(e, s)$-continuous.*

**Example 11** *Let $X = \{a, b, c, d\}$ and $\tau = \{\varnothing, X, \{a, d\}, \{c\}, \{a, c, d\}\}$. Let $f : X \to X$ be a function defined by $f(a) = d$, $f(b) = a$, $f(c) = b$, $f(d) = c$. Then, $f$ is $(\delta$-pre$, s)$-continuous but it is not $(a, s)$-continuous.*

**Example 12** *Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. Then the identity function $i : X \to X$ is $(e, s)$-continuous but it is not $(\delta$-pre$, s)$-continuous.*

**Definition 13** *A function $f : X \to Y$ is said to be:*

*(1) $e^*$-continuous [11] if $f^{-1}(N)$ is $e^*$-open in $X$ for every open set $N$ of $Y$.*

*(2) almost $e^*$-continuous [13] (resp. almost e-continuous [13], almost a-continuous [13]) if $f^{-1}(N)$ is $e^*$-open (resp. e-open, a-open) in $X$ for every regular open set $N$ of $Y$.*

A topological space $(X, \tau)$ is said to be extremally disconnected [4] if the closure of every open set of $X$ is open in $X$.

**Theorem 14** *Let $(Y, \sigma)$ be extremally disconnected. The following are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$:*

*(1) $f$ is $(e^*, s)$-continuous,*

*(2) $f$ is almost $e^*$-continuous.*

**Proof.** $(1) \Rightarrow (2)$ : Let $U \in RO(Y)$. Since $Y$ is extremally disconnected, by Lemma 5.6 of [20] $U$ is clopen. Since $U$ is regular closed, then $f^{-1}(U)$ is $e^*$-open. Hence, $f$ is almost $e^*$-continuous.

$(2) \Rightarrow (1)$ : Let $S \in RC(Y)$. Since $Y$ is extremally disconnected, $S$ is regular open. Thus, $f^{-1}(S)$ is $e^*$-open and hence $f$ is $(e^*, s)$-continuous. ■

**Definition 15** *A space $(X, \tau)$ is called $e^*$-$T_{1/2}$ [13] if every $e^*$-closed set is $\delta$-closed.*

**Theorem 16** *Let $f : X \to Y$ be a function from an $e^*$-$T_{1/2}$ space $X$ to a topological space $Y$. The following are equivalent:*

*(1) $f$ is $(e^*, s)$-continuous,*

*(2) $f$ is $(e, s)$-continuous,*

*(3) $f$ is $(\delta$-semi$, s)$-continuous,*

*(4) $f$ is $(\delta$-pre$, s)$-continuous,*

*(5) $f$ is $(a, s)$-continuous,*

*(6) $f$ is almost contra-super-continuous.*

**Theorem 17** *Let $Y$ be a regular space and $f : X \to Y$ be a function. If $f$ is $(e^*, s)$-continuous, then $f$ is $e^*$-continuous.*

**Proof.** Let $x \in X$ and $N$ be an open set of $Y$ containing $f(x)$. Since $Y$ is regular, there exists an open set $G$ in $Y$ containing $f(x)$ such that $cl(G) \subset N$. Since $f$ is $(e^*, s)$-continuous, there exists $U \in e^*O(X, x)$ such that $f(U) \subset cl(G)$. Thus, $f(U) \subset cl(G) \subset N$ and hence $f$ is $e^*$-continuous. ■

**Definition 18** *A function $f : X \to Y$ is said to be $e^*$-irresolute [13] (resp. e-irresolute, a-irresolute) if $f^{-1}(N)$ is $e^*$-open (resp. e-open, a-open) in $X$ for every $N \in e^*O(Y)$ (resp. $N \in eO(Y)$, $N \in aO(Y)$).*

**Theorem 19** *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then, the following properties hold:*

*(1) If $f$ is $e^*$-irresolute and $g$ is $(e^*, s)$-continuous, then $g \circ f$ is $(e^*, s)$-continuous.*

*(2) If $f$ is $(e^*, s)$-continuous and $g$ is contra R-map, then $g \circ f$ is almost $e^*$-continuous.*

*(3) If $f$ is $e^*$-continuous and $g$ is almost contra-super-continuous, then $g \circ f$ is $(e^*, s)$-continuous.*

*(4) If $f$ is $e^*$-irresolute and $g$ is $e^*$-irresolute, then $g \circ f$ is $e^*$-irresolute.*

*(5) If $f$ is almost $e^*$-continuous and $g$ is contra R-map, then $g \circ f$ is $(e^*, s)$-continuous.*

**Theorem 20** *The following are equivalent for a function $f : X \to Y$:*

$(a)$ *$f$ is $(e^*, s)$-continuous (resp. $(e, s)$-continuous, $(a, s)$-continuous),*

$(b)$ *the inverse image of a regular closed set of $Y$ is $e^*$-open (resp. e-open, a-open),*

$(c)$ *$f(e^*\text{-}cl(U)) \subset r\text{-}ker(f(U))$ (resp. $f(e\text{-}cl(U)) \subset r\text{-}ker(f(U))$, $f(a\text{-}cl(U)) \subset r\text{-}ker(f(U))$) for every $U \subset X$,*

$(d)$ *$e^*\text{-}cl(f^{-1}(N)) \subset f^{-1}(r\text{-}ker(N))$ (resp. $e\text{-}cl(f^{-1}(N)) \subset f^{-1}(r\text{-}ker(N))$, $a\text{-}cl(f^{-1}(N)) \subset f^{-1}(r\text{-}ker(N))$) for every $N \subset Y$,*

$(e)$ *for each $x \in X$ and each $N \in SO(Y, f(x))$, there exists an $e^*$-open (resp. e-open, a-open) set $U$ in $X$ containing $x$ such that $f(U) \subset cl(N)$,*

$(f)$ *$f(e^*\text{-}cl(P)) \subset \theta\text{-}s\text{-}cl(f(P))$ (resp. $f(e\text{-}cl(P)) \subset \theta\text{-}s\text{-}cl(f(P))$, $f(a\text{-}cl(P)) \subset \theta\text{-}s\text{-}cl(f(P))$) for every $P \subset X$,*

$(g)$ *$e^*\text{-}cl(f^{-1}(R)) \subset f^{-1}(\theta\text{-}s\text{-}cl(R))$ (resp. $e\text{-}cl(f^{-1}(R)) \subset f^{-1}(\theta\text{-}s\text{-}cl(R))$, $a\text{-}cl(f^{-1}(R)) \subset f^{-1}(\theta\text{-}s\text{-}cl(R))$) for every $R \subset Y$,*

$(h)$ *$e^*\text{-}cl(f^{-1}(N)) \subset f^{-1}(\theta\text{-}s\text{-}cl(N))$ (resp. $e\text{-}cl(f^{-1}(N)) \subset f^{-1}(\theta\text{-}s\text{-}cl(N))$, $a\text{-}cl(f^{-1}(N)) \subset f^{-1}(\theta\text{-}s\text{-}cl(N))$) for every open subset $N$ of $Y$,*

$(i)$ *$e^*\text{-}cl(f^{-1}(N)) \subset f^{-1}(s\text{-}cl(N))$ (resp. $e\text{-}cl(f^{-1}(N)) \subset f^{-1}(s\text{-}cl(N))$, $a\text{-}cl(f^{-1}(N)) \subset f^{-1}(s\text{-}cl(N))$) for every open subset $N$ of $Y$,*

$(j)$ *$e^*\text{-}cl(f^{-1}(N)) \subset f^{-1}(int(cl(N)))$ (resp. $e\text{-}cl(f^{-1}(N)) \subset$ $\subset f^{-1}(int(cl(N)))$, $a\text{-}cl(f^{-1}(N)) \subset f^{-1}(int(cl(N)))$) for every open subset $N$ of $Y$,*

$(k)$ *the inverse image of a $\theta$-semi-open set of $Y$ is $e^*$-open (resp. e-open, a-open),* *(l)* $f^{-1}(N) \subset e^*\text{-}int(f^{-1}(cl(N)))$ *(resp.* $f^{-1}(N) \subset e\text{-}int(f^{-1}(cl(N)))$, $f^{-1}(N) \subset a\text{-}int(f^{-1}(cl(N))))$ *for every* $N \in SO(Y)$,

*(m) the inverse image of a* $\theta$*-semi-closed set of* $Y$ *is* $e^*$*-closed (resp. e-closed, a-closed),*

*(n)* $f^{-1}(int(cl(N)))$ *is* $e^*$*-closed (resp. e-closed, a-closed) for every open subset* $N$ *of* $Y$,

*(o)* $f^{-1}(cl(int(F)))$ *is* $e^*$*-open (resp. e-open, a-open) for every closed subset* $F$ *of* $Y$,

*(p)* $f^{-1}(cl(U))$ *is* $e^*$*-open (resp. e-open, a-open) in* $X$ *for every* $U \in \beta O(Y)$,

*(r)* $f^{-1}(cl(U))$ *is* $e^*$*-open (resp. e-open, a-open) in* $X$ *for every* $U \in SO(Y)$,

*(s)* $f^{-1}(int(cl(U)))$ *is* $e^*$*-closed (resp. e-closed, a-closed) in* $X$ *for every* $U \in PO(Y)$.

**Proof.** We prove only for $(e^*, s)$-continuity, the proofs for $(e, s)$-continuity and $(a, s)$-continuity being entirely analogous.

$(a) \Leftrightarrow (b)$ : Obvious.

$(b) \Rightarrow (c)$ : Let $U \subset X$. Let $y \notin$r-$ker(f(U))$. There exists a regular closed set $F$ containing $y$ such that $f(U) \cap F = \varnothing$. We have $U \cap f^{-1}(F) = \varnothing$ and $e^*\text{-}cl(U) \cap f^{-1}(F) = \varnothing$. Thus, $f(e^*\text{-}cl(U)) \cap F = \varnothing$ and $y \notin f(e^*\text{-}cl(U))$. Hence, $f(e^*\text{-}cl(U)) \subset$r-$ker(f(U))$.

$(c) \Rightarrow (d)$ : Let $N \subset Y$. By $(c)$, $f(e^*\text{-}cl(f^{-1}(N))) \subset$r-$ker(N)$. Thus, $e^*\text{-}cl(f^{-1}(N)) \subset f^{-1}($r-$ker(N))$.

$(d) \Rightarrow (a)$ : Let $N \in RO(Y)$. By Lemma 1, $e^*\text{-}cl(f^{-1}(N) \subset f^{-1}($r-$ker(N)) = f^{-1}(N)$ and $e^*\text{-}cl((f^{-1}(N)) = f^{-1}(N)$. Hence, $f^{-1}(N)$ is $e^*$-closed in $X$.

$(e) \Rightarrow (f)$ : Let $P \subset X$ and $x \in e^*\text{-}cl(P)$ and $G \in SO(Y, f(x))$. By $(e)$, there exists $U \in e^*O(X, x)$ such that $f(U) \subset cl(G)$. Since $x \in e^*\text{-}cl(P)$, $U \cap P \neq \emptyset$ and $\emptyset \neq f(U) \cap f(P) \subset cl(G) \cap f(P)$. Thus, $f(x) \in \theta\text{-}s\text{-}cl(f(P))$ and hence $f(e^*\text{-}cl(P)) \subset \theta\text{-}s\text{-}cl(f(P))$.

$(f) \Rightarrow (g)$ : Let $R \subset Y$. We have $f(e^*\text{-}cl(f^{-1}(R))) \subset \theta\text{-}s\text{-}cl(f(f^{-1}(R))) \subset \theta\text{-}s\text{-}cl(R)$ and $e^*\text{-}cl(f^{-1}(R)) \subset f^{-1}(\theta\text{-}s\text{-}cl(R))$.

$(g) \Rightarrow (e)$ : Let $N \in SO(Y, f(x))$. Since $cl(N) \cap (Y \backslash cl(N)) = \emptyset$, then $f(x) \notin \theta\text{-}s\text{-}cl(Y \backslash cl(N))$ and $x \notin f^{-1}(\theta\text{-}s\text{-}cl(Y \backslash cl(N)))$. By $(g)$, $x \notin e^*\text{-}cl(f^{-1}(Y \backslash cl(N)))$ and hence there exists $U \in e^*O(X, x)$ such that $U \cap f^{-1}(Y \backslash cl(N)) = \varnothing$ and $f(U) \cap (Y \backslash cl(N)) = \varnothing$. It follows that $f(U) \subset cl(N)$.

$(g) \Rightarrow (h)$ : Obvious.

$(h) \Rightarrow (i)$ : Since $\theta\text{-}s\text{-}cl(N) = s\text{-}cl(N)$ for an open set $N$, it is obvious.

$(i) \Rightarrow (j)$ : It follows from Lemma 4.

$(j) \Rightarrow (a)$ : Let $N \in RO(Y)$. By $(j)$, $e^*\text{-}cl(f^{-1}(N)) \subset f^{-1}(int(cl(N))) = f^{-1}(N)$. Hence, $f^{-1}(N)$ is $e^*$-closed and hence $f$ is $(e^*, s)$-continous.

$(b) \Rightarrow (k)$ : Since any $\theta$-semi-open set is a union of regular closed sets, it holds.

$(k) \Rightarrow (e)$ : Let $x \in X$ and $N \in SO(Y, f(x))$. Since $cl(N)$ is $\theta$-semi-open in $Y$, there exists an $e^*$-open set $U$ such that $x \in U \subset f^{-1}(cl(N))$. Hence, $f(U) \subset cl(N)$.

$(e) \Rightarrow (l)$ : Let $N \in SO(Y)$ and $x \in f^{-1}(N)$. We have $f(x) \in N$ and there exists an $e^*$-open set $U$ in $X$ containing $x$ such that $f(U) \subset cl(N)$. We have $x \in U \subset f^{-1}(cl(N))$ and hence $x \in e^*\text{-}int(f^{-1}(cl(N)))$. Thus, $f^{-1}(N) \subset e^*\text{-}int(f^{-1}(cl(N)))$.

$(l) \Rightarrow (b)$ : Let $F$ be any regular closed set of $Y$. Since $F \in SO(Y)$, then $f^{-1}(F) \subset e^*\text{-}int(f^{-1}(F))$. This shows that $f^{-1}(F)$ is $e^*$-open in $X$.

$(k) \Leftrightarrow (m)$ : It is obvious.

$(a) \Leftrightarrow (n)$ : Let $N$ be an open subset of $Y$. Since $int(cl(N))$ is regular open, $f^{-1}(int(cl(N)))$ is $e^*$-closed. The converse is similar.

$(b) \Leftrightarrow (o)$ : It is obvious.

$(b) \Rightarrow (p)$ : Let $U \in \beta O(Y)$. By Theorem 2.4 [2], $cl(U)$ is regular closed and hence $f^{-1}(cl(U)) \in e^*O(X)$.

$(p) \Rightarrow (r)$ : Since $SO(Y) \subset \beta O(Y)$, it is obvious.

$(r) \Rightarrow (s)$ : Let $U \in PO(Y)$. Since $Y \backslash int(cl(U))$ is regular closed and hence it is semiopen, we have $X \backslash f^{-1}(int(cl(U)) = f^{-1}(Y \backslash int(cl(U))) = f^{-1}(cl(Y \backslash int(cl(U)))) \in e^*O(X)$. Thus, $f^{-1}(int(cl(U)))$ is $e^*$-closed.

$(s) \Rightarrow (a)$ : Let $U \in RO(Y)$ Then $U \in PO(Y)$ and hence $f^{-1}(U) = f^{-1}(int(cl(U)))$ is $e^*$-closed in $X$. ■

**Lemma 21** *([19]) The following properties hold for a subset $P$ of a space $X$:*

*(1) $\alpha\text{-}cl(P) = cl(P)$ for every $P \in \beta O(X)$,*

*(2) $p\text{-}cl(P) = cl(P)$ for every $P \in SO(X)$.*

**Corollary 22** *The following are equivalent for a function $f : X \to Y$:*

*(1) $f$ is $(e^*, s)$-continuous (resp. $(e, s)$-continuous, $(a, s)$-continuous),*

*(2) $f^{-1}(\alpha\text{-}cl(N))$ is $e^*$-open (resp. e-open, a-open) in $X$ for every $N \in \beta O(Y)$,*

*(3) $f^{-1}(p\text{-}cl(N))$ is $e^*$-open (resp. e-open, a-open) in $X$ for every $N \in SO(Y)$,*

*(4) $f^{-1}(s\text{-}cl(N))$ is $e^*$-closed (resp. e-closed, a-closed) in $X$ for every $N \in PO(Y)$,*

*(5)* $e^*$-$cl(f^{-1}(R)) \subset f^{-1}(\theta$-$s$-$cl(R))$ *(resp.* $e$-$cl(f^{-1}(R)) \subset f^{-1}(\theta$-$s$-$cl(R))$, $a$-$cl(f^{-1}(R)) \subset f^{-1}(\theta$-$s$-$cl(R)))$ *for every* $R \in SO(Y)$,

*(6)* $e^*$-$cl(f^{-1}(R)) \subset f^{-1}(\theta$-$s$-$cl(R))$ *(resp.* $e$-$cl(f^{-1}(R)) \subset f^{-1}(\theta$-$s$-$cl(R))$, $a$-$cl(f^{-1}(R)) \subset f^{-1}(\theta$-$s$-$cl(R)))$ *for every* $R \in PO(Y)$,

*(7)* $e^*$-$cl(f^{-1}(R)) \subset f^{-1}(\theta$-$s$-$cl(R))$ *(resp.* $e$-$cl(f^{-1}(R)) \subset f^{-1}(\theta$-$s$-$cl(R))$, $a$-$cl(f^{-1}(R)) \subset f^{-1}(\theta$-$s$-$cl(R)))$ *for every* $R \in \beta O(Y)$.

**Proof.** It follows from Lemma 21, Lemma 4 and Theorem 20. ■

**Definition 23** *A subset* $P$ *of a topological space* $X$ *is said to be* $e^*$*-dense (resp. e-dense) in* $X$ *if* $e^*$-$cl(P) = X$ *(resp.* $e$-$cl(P) = X$*).*

A space $X$ is said to be s-Urysohn [3] if for each pair of distinct points $x$ and $y$ in $X$, there exist $M \in SO(X, x)$ and $N \in SO(X, y)$ such that $cl(M) \cap cl(N) = \emptyset$.

**Lemma 24** *([7]) Let* $f : X \to Y$ *be a function. If* $f$ *is almost contra-super-continuous, then for each* $x \in X$ *and for each* $V \in SO(Y, f(x))$*, there exists a* $\delta$*-open set* $U$ *in* $X$ *containing* $x$ *such that* $f(U) \subset cl(V)$.

**Lemma 25** *([13]) Let* $X$ *be a space and* $A, B \subset X$*. If* $A \in \delta O(X)$ *and* $B \in e^*O(X)$ *(resp.* $B \in eO(X)$*), then* $A \cap B \in e^*O(X)$ *(resp.* $A \cap B \in eO(X)$*).*

**Theorem 26** *Let* $f,\ g : X \to Y$ *be functions. If* $f$ *is* $(e^*, s)$*-continuous (resp.* $(e, s)$*-continuous) and* $g$ *is almost contra-super-continuous and* $Y$ *is s-Urysohn, then* $P = \{x \in X : f(x) = g(x)\}$ *is* $e^*$*-closed (resp. e-closed) in* $X$.

**Proof.** Let $x \in X \backslash P$. We have $f(x) \neq g(x)$. Since $Y$ is s-Urysohn, there exist $M \in SO(Y, f(x))$ and $N \in SO(Y, g(x))$ such that $cl(M) \cap cl(N) = \emptyset$. Since $f$ is $(e^*, s)$-continuous and $g$ is almost contra-super-continuous, there exist an $e^*$-open set $K$ and a $\delta$-open set $L$ containing $x$ such that $f(K) \subset cl(M)$ and $g(L) \subset cl(N)$. Thus, $K \cap L = S \in e^*O(X)$, $f(S) \cap g(S) = \emptyset$ and hence $x \notin e^*$-$cl(P)$. Hence, $P$ is $e^*$-closed in $X$. ■

**Theorem 27** *Let* $X$ *and* $Y$ *be topological spaces. If* $Y$ *is s-Urysohn,* $f : X \to Y$ *and* $g : X \to Y$ *are* $(e^*, s)$*-continuous (resp.* $(e, s)$*-continuous) and almost contra-super-continuous functions, respectively and* $f = g$ *on* $e^*$*-dense (resp. e-dense) set* $P \subset X$*, then* $f = g$ *on* $X$.

**Proof.** Let $f$ and $g$ be $(e^*, s)$-continuous and almost contra-super-continuous functions, respectively and $Y$ be s-Urysohn. Then $R = \{x \in X : f(x) = g(x)\}$ is $e^*$-closed in $X$. Since $P \subset R$ and $P$ is $e^*$-dense set in $X$, $X = e^*\text{-}cl(P) \subset e^*\text{-}cl(R) = R$. Thus, $f = g$ on $X$. ■

**Definition 28** *Let $X$ be a topological space and $P \subset X$. The $e^*$-frontier (resp. e-frontier, a-frontier) of $P$ is given by $e^*\text{-}fr(P) = e^*\text{-}cl(P) \cap e^*\text{-}cl(X\backslash P)$ (resp. $e\text{-}fr(P) = e\text{-}cl(P) \cap e\text{-}cl(X\backslash P)$, $a\text{-}fr(P) = a\text{-}cl(P) \cap a\text{-}cl(X\backslash P)$).*

**Theorem 29** *Let $f : X \to Y$ be a function. Then $f$ is not $(e^*, s)$-continuous (resp. $(e, s)$-continuous, $(a, s)$-continuous) at $x$ if and only if $x \in e^*\text{-}fr(f^{-1}(F))$ (resp. $x \in e\text{-}fr(f^{-1}(F))$, $x \in a\text{-}fr(f^{-1}(F))$) for some $F \in RC(Y, f(x))$.*

**Proof.** $(\Rightarrow)$ : Suppose that $f$ is not $(e^*, s)$-continuous at $x$. There exists $F \in RC(Y, f(x))$ such that $f(U) \nsubseteq F$ for every $U \in e^*O(X, x)$. For every $U \in e^*O(X, x)$, we have $f(U) \cap (Y\backslash F) \neq \varnothing$. Thus, $U \cap (X\backslash f^{-1}(F)) \neq \varnothing$ for every $U \in e^*O(X, x)$ and hence $x \in e^*\text{-}cl(X\backslash f^{-1}(F))$. Since $x \in f^{-1}(F)$, $x \in e^*\text{-}fr(f^{-1}(F))$.

$(\Leftarrow)$ : Let $x \in X$. Suppose that there exists $F \in RC(Y, f(x))$ such that $x \in e^*\text{-}fr(f^{-1}(F))$ and that $f$ is $(e^*, s)$-continuous at $x$. There exists an $e^*$-open set $U$ such that $x \in U$ and $U \subset f^{-1}(F)$. Thus, $x \notin e^*\text{-}cl(X\backslash f^{-1}(F))$. This is a contradiction. Hence, $f$ is not $(e^*, s)$-continuous at $x$. ■

## 3 Further properties

**Definition 30** *A subset $S$ of a space $X$ is said to be $e^*$-compact (resp. e-compact, a-compact) relative to $X$ if for every cover $\{P_i : i \in I\}$ of $S$ by $e^*$-open (resp. e-open, a-open) sets of $X$, there exists a finite subset $I_0$ of $I$ such that $S \subset \cup\{P_i : i \in I_0\}$. A space $X$ said to be $e^*$-compact [13] (resp. e-compact [13], a-compact [13]) if every $e^*$-open (resp. e-open, a-open) cover of $X$ has a finite subcover.*

**Theorem 31** *Every $e^*$-closed (resp. e-closed, a-closed) subset $P$ of an $e^*$-compact (resp. e-compact, a-compact) space $X$ is $e^*$-compact (resp. e-compact, a-compact) relative to $X$.*

**Proof.** Let $P \subset X$ be $e^*$-closed and $X$ be an $e^*$-compact space. Let $\{M_i : i \in I\}$ be a cover of $P$ by $e^*$-open subsets of $X$. This implies that $P \subset \underset{i\in I}{\cup} M_i$ and $(X \backslash P) \cup (\underset{i\in I}{\cup} M_i) = X$. Since $X$ is $e^*$-compact, there exists a finite subset $I_0$ of $I$ such that $(X \backslash P) \cup (\underset{i\in I_0}{\cup} M_i) = X$. Thus $P \subset \underset{i\in I_0}{\cup} M_i$ and hence $P$ is $e^*$-compact relative to $X$. ■

A space $X$ said to be S-closed [25] if every regular closed cover of $X$ has a finite subcover.

**Theorem 32** *The surjective $(e^*, s)$-continuous (resp. $(e, s)$-continuous, $(a, s)$-continuous) image of an $e^*$-compact (resp. e-compact, a-compact) space is S-closed.*

**Proof.** Let $X$ be an $e^*$-compact space and $f : X \to Y$ be a surjective $(e^*, s)$-continuous function. Let $\{M_i : i \in I\}$ be a cover of $Y$ by regular closed sets. Since $f$ is $(e^*, s)$-continuous, then $\{f^{-1}(M_i) : i \in I\}$ is a cover of $X$ by $e^*$-open sets. Since $X$ is $e^*$-compact, there exists a finite subset $I_0$ of $I$ such that $X = \underset{i\in I_0}{\cup} f^{-1}(M_i)$. Since $f$ is surjective, $Y = \underset{i\in I_0}{\cup} M_i$. Thus, $Y$ is S-closed. ■

**Theorem 33** *If $f : X \to Y$ is $e^*$-irresolute (resp. e-irresolute, a-irresolute) and $P \subset X$ is $e^*$-compact (resp. e-compact, a-compact) relative to $X$, then its image $f(P)$ is $e^*$-compact (resp. e-compact, a-compact) relative to $Y$.*

**Proof.** It is similar to that of Theorem 32. ■

**Definition 34** *A space $X$ is said to be $e^*$-$T_1$ (resp. e-$T_1$, a-$T_1$) if for each pair of distinct points in $X$, there exist $e^*$-open (resp. e-open, a-open) sets $M$ and $N$ containing $x$ and $y$, respectively, such that $y \notin M$ and $x \notin N$.*

A space $X$ is said to be weakly Hausdorff [23] if each element of $X$ is an intersection of regular closed sets.

**Theorem 35** *Let $f : X \to Y$ be a function. If $f$ is a $(e^*, s)$-continuous (resp. $(e, s)$-continuous, $(a, s)$-continuous) injection and $Y$ is weakly Hausdorff, then $X$ is $e^*$-$T_1$ (resp. e-$T_1$, a-$T_1$).*

**Proof.** For $x \neq y$ in $X$, there exist $P$, $R \in RC(Y)$ such that $f(x) \in P$, $f(y) \notin P$, $f(x) \notin R$ and $f(y) \in R$. Since $f$ is $(e^*, s)$-continuous, $f^{-1}(P)$ and $f^{-1}(R)$ are $e^*$-open subsets of $X$ such that $x \in f^{-1}(P)$, $y \notin f^{-1}(P)$, $x \notin f^{-1}(R)$ and $y \in f^{-1}(R)$. Thus, $X$ is $e^*$-$T_1$. ■

**Definition 36** *A space $X$ is said to be $e^*$-$T_2$ [13] (resp. e-$T_2$ [13], a-$T_2$ [13]) if for each pair of distinct points $x$ and $y$ in $X$, there exist $M \in e^*O(X,x)$ (resp. $M \in eO(X,x)$, $M \in aO(X,x)$) and $N \in e^*O(X,y)$ (resp. $N \in eO(X,y)$, $N \in aO(X,y)$) such that $M \cap N = \emptyset$.*

**Theorem 37** *Let $f : X \to Y$ be a function. If $f$ is a $(e^*, s)$-continuous (resp. $(e, s)$-continuous, $(a, s)$-continuous) injection and $Y$ is s-Urysohn, then $X$ is $e^*$-$T_2$ (resp. e-$T_2$, a-$T_2$).*

**Proof.** Let $Y$ be s-Urysohn. For any distinct points $x$ and $y$ in $X$, $f(x) \neq f(y)$. Since $Y$ is s-Urysohn, there exist $P \in SO(Y, f(x))$ and $R \in SO(Y, f(y))$ such that $cl(P) \cap cl(R) = \emptyset$. Since $f$ is a $(e^*, s)$-continuous, there exist $e^*$-open sets $A$ and $B$ in $X$ containing $x$ and $y$, respectively, such that $f(A) \subset cl(P)$ and $f(B) \subset cl(R)$ such that $A \cap B = \emptyset$. Thus, $X$ is $e^*$-$T_2$. ■

## References


[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, $\beta$-open sets and $\beta$-continuous mappings, Bull. Fac. Sci. Assiut Univ. 12 (1983) 77-90.

[2] D.Andrijevic, Semipreopen sets, Mat. Vesnik, 38 (1986), 24–32.

[3] S. P. Arya and M. P. Bhamini, Some generalizations of pairwise Urysohn spaces, Indian J. Pure Appl. Math. 18 (1987) 1088-1093.

[4] N. Bourbaki, General topology, Part I, Addison Wesley, Reading, Mass 1996.

[5] S. G. Crossley and S. K. Hildebrand, Semi-closure, Texas J. Sci. 22 (1971) 99-112.

[6] E. Ekici, On contra R-continuity and a weak form, Indian J. Math., 46 (2-3) (2004), 267-281.

[7] E. Ekici, Almost contra-super-continuous functions, Studii si Cercetari Stiintifice, Seria: Matematica, Univ. din Bacau, 14 (2004), 31-42.

[8] E. Ekici, ($\delta$-pre,s)-continuous functions, Bull. Malaysian Math. Sci. Soc., 27 (2) (2004), 237-251.

[9] E. Ekici, Another form of contra-continuity, Kochi Journal of Mathematics, 1 (2006), 21-29.

[10] E. Ekici, On $e$-open sets, $\mathcal{DP}^*$-sets and $\mathcal{DPE}^*$-sets and decompositions of continuity (submitted).

[11] E. Ekici, On $e^*$-open sets and $(\mathcal{D}, \mathcal{S})^*$-sets and decompositions of continuous functions (submitted).

[12] E. Ekici, On $a$-open sets, $\mathcal{A}^*$-sets and decompositions of continuity and super-continuity (submitted).

[13] E. Ekici, Some weak forms of $\delta$-continuity and $e^*$-first-countable spaces (submitted).

[14] J. E. Joseph and M. H. Kwack, On S-closed spaces, Proc. Amer. Math. Soc., 80 (1980), 341-348.

[15] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly 70 (1963) 36-41.

[16] G.Di Maio and T. Noiri, On s-closed spaces, Indian J. Pure Appl. Math., 18 (1987), 226-233.

[17] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt, 53 (1982), 47-53.

[18] O. Njastad, On some classes of nearly open sets, Pacific J. Math., 15 (1965), 961-970.

[19] T. Noiri, On almost continuous functions, Indian J. Pure Appl. Math., 20 (1989), 571–576.

[20] M. C. Pal and P. Bhattacharyya, Faint precontinuous functions, Soochow J. Math., 21 (1995), 273–289.

[21] J. H. Park, B. Y. Lee and M. J. Son, On $\delta$-semiopen sets in topological space, J. Indian Acad. Math., 19 (1) (1997), 59-67.

[22] S. Raychaudhuri and M. N. Mukherjee, On $\delta$-almost continuity and $\delta$-preopen sets, Bull. Inst. Math. Acad. Sinica, 21 (1993), 357-366.

[23] T. Soundararajan, Weakly Hausdorff spaces and the cardinality of topological spaces, in: General Topology and its relation to Modern Analysis and Algebra. III, Proc. Conf. Kanpur, 1968, Academia, Prague 1971, pp. 301-306.

[24] M. H. Stone, Applications of the theory of Boolean rings to general topology, TAMS 41 (1937) 375-381.

[25] T. Thompson, S-closed spaces, Proc. Amer. Math. Soc. 60 (1976) 335-338.

[26] N. V. Veličko, H-closed topological spaces, Amer. Math. Soc. Transl., 78 (1968) 103-118.

Department of Mathematics, Canakkale Onsekiz Mart University,
Terzioglu Campus, 17020 Canakkale, Turkey
*E-mail*: eekici@comu.edu.tr