Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **21:2** (2007), 45–53

COINCIDENCE POINTS AND INVARIANT APPROXIMATION RESULTS ON q-NORMED SPACES

Hemant Kumar Nashine

Abstract

We present coincidence points results for multivalued f- nonexpansive mappings in the setting of nonstarshaped domain of q-normed space which is not necessarily a locally convex space. As application, an invariant approximation result is also obtained. Our results improve and extend the results of Bano, Khan and Latif [1], Hussain [4], Latif and Tweddle [7], Rhoades [11], Sahab, Khan and Sessa [13], Shahzad [14] and Singh [15] in the setting of nonstarshaped domain.

1 Introduction and preliminaries

Let $\mathcal{X}$ be a linear space. A q-norm on $\mathcal{X}$ is a real-valued function $\|.\|_q$ on $\mathcal{X}$ with $0 < q \leq 1$, satisfying the following conditions:

(a) $\|x\|_q \geq 0$ and $\|x\|_q = 0$ iff $x = 0$,

(b) $\|\lambda x\|_q = |\lambda|^q \|x\|_q$,

(c) $\|x + y\|_q \leq \|x\|_q + \|y\|_q$,

for all $x, y \in \mathcal{X}$ and all scalars λ. The pair $(\mathcal{X}, \|.\|_q)$ is called a q-normed space. It is a metric space with $d_q(x, y) = \|x - y\|_q$ for all $x, y \in \mathcal{X}$, defining a translation invariant metric d_q on $\mathcal{X}$. If $q = 1$, we obtain the concept of a normed linear space. It is well-known that the topology of every housdorff locally bounded topological linear space is given by some q-norm, $0 < q \leq 1$. The spaces l_q and $\mathcal{L}_q[0, 1]$, $0 < q \leq 1$ are q-normed space. A q-normed space

2000 *Mathematics Subject Classification.* 41A50, 47H10, 54H25.
*Key words and phrases.*Coincidence point, invariant approximation, q-normed space, multivalued f-nonexpansive mapping.
Received: August 19, 2006

is not necessarily a locally convex space. Recall that, if $\mathcal{X}$ is a topological linear space, then its continuous dual space $\mathcal{X}^*$ is said to separate the points of $\mathcal{X}$, if for each $x \neq 0$ in $\mathcal{X}$, there exists an $g \in \mathcal{X}^*$ such that $g(x) \neq 0$. In this case the weak topology on $\mathcal{X}$ is well-defined. We mention that, if $\mathcal{X}$ is not locally convex, then $\mathcal{X}^*$ need not separates the points of $\mathcal{X}$. For example, if $\mathcal{X} = \mathcal{L}_q[0,1]$, $0 < q < 1$, then $\mathcal{X}^* = \{0\}$ ([12, page 36 and 37]). However, there are some non-locally convex spaces (such as the q-normed space l_q, $0 < q < 1$) whose dual separates the points [5].

Let $\mathcal{L}_q$, $0 < q \leq 1$ be the space of all measurable functions $f(t)$ on $\mathcal{I} = [a,b]$ with $\int_a^b |f(t)|^q dt < \infty$ (we identify functions which are equal almost everywhere). For all $f \in \mathcal{L}_q$, $0 < q \leq 1$, let the function $\|f\|_q$ be defined by

$$\|f\|_q = (\int_a^b |f(t)|^q dt)^{\frac{1}{q}}. \tag{1.1}$$

This expression is an example of a quasinorm on a topological linear space [5].

Let $\mathcal{X}$ be a q-normed space and $\mathcal{M}$ a nonempty subset of $\mathcal{X}$. We denote by $2^{\mathcal{X}}, \mathcal{C}(\mathcal{M}), \mathcal{CB}(\mathcal{X}), \mathcal{K}(\mathcal{X})$ the collection of all nonempty, nonempty closed, nonempty closed bounded and nonempty compact subsets of $\mathcal{X}$, respectively. The Hausdorff metric $\mathcal{H}_q$ on $\mathcal{C}(\mathcal{X})$ induced by the q-norm of $\mathcal{X}$ is defined by

$$\mathcal{H}_q(\mathcal{A},\mathcal{B}) = \max\{\sup_{a\in\mathcal{A}} d_q(a,\mathcal{B}), \sup_{b\in\mathcal{B}} d_q(\mathcal{A},b)\} \tag{1.2}$$

for all $\mathcal{A},\mathcal{B} \in \mathcal{C}(\mathcal{X})$, where $d_q(x,\mathcal{B}) = \inf\{\|x-y\|_q : y \in \mathcal{B}\}$ for each $x \in \mathcal{X}$. Let $\mathcal{M}$ be a subset of q-normed space $\mathcal{X}$ for each $x \in \mathcal{X}$, define

$$\mathcal{P}_{\mathcal{M}}(x) = \{y \in \mathcal{M} : \|x-y\|_q = d_q(x,\mathcal{M})\},$$

the set of the best $\mathcal{M}$-approximants to x.
The set $\mathcal{P}_{\mathcal{M}}(x)$ is always a bounded subset of $\mathcal{X}$ and it is closed or convex if $\mathcal{M}$ is closed or convex (see [2]).

Let $f : \mathcal{M} \to \mathcal{X}$ be a single-valued map. A multivalued map $\mathcal{T} : \mathcal{M} \to \mathcal{C}(\mathcal{X})$ is said to be a f-contraction if for a fixed constant k, $0 \leq k < 1$ and for all $x, y \in \mathcal{X}$.

$$\mathcal{H}_q(\mathcal{T}(x),\mathcal{T}(y)) \leq k\|f(x) - f(y)\|_q.$$

If $k = 1$ in the above inequality then $\mathcal{T}$ is called f-nonexpansive. Indeed, if $f = \mathcal{I}$ (the identity map on $\mathcal{X}$), then each f-contraction is a contraction. An element $x \in \mathcal{X}$ is called a fixed point of multivalued map $\mathcal{T}$ if $x \in \mathcal{T}(x)$.

An element $x \in \mathcal{X}$ is called a coincidence point (common fixed point) of f and $\mathcal{T}$ if $f(x) \in \mathcal{T}(x)$ $(x = f(x) \in \mathcal{T}(x))$. We denote by $\mathcal{F}(\mathcal{T})$ the set of fixed points of $\mathcal{T}$ and by $\mathcal{C}(f \cap \mathcal{T})$ the set of coincidence points of f and $\mathcal{T}$.

Let $\mathcal{M}$ a nonempty subset of a q-normed space $\mathcal{X}$. The pair $\{f, \mathcal{T}\}$ is called (1) commuting if $f\mathcal{T}x = \mathcal{T}fx$ for all $x \in \mathcal{M}$; (2) $\mathcal{R}-$weakly commuting if for all $x \in \mathcal{M}$, $f\mathcal{T}x \in \mathcal{C}(\mathcal{M})$ and there exists $\mathcal{R} > 0$ such that $\mathcal{H}_q(\mathcal{T}fx, f\mathcal{T}x) \leq \mathcal{R}\|\mathcal{T}x - fx\|_q$. The set $\mathcal{M}$ is $p-$starshaped with $p \in \mathcal{F}(f)$ if the segment $[p, x] = \{(1-k)p + kx, 0 \leq k \leq 1\}$ joining p to x, is contained in $\mathcal{M}$ for all $x \in \mathcal{M}$. Suppose $\mathcal{M}$ is $p-$starshaped with $p \in \mathcal{F}(f)$ and is both $\mathcal{T}-$ and f-invariant. Then $\mathcal{T}$ and f are called $\mathcal{R}$-subweakly commuting on $\mathcal{M}$ [4, 14] if for all $x \in \mathcal{M}$, $f\mathcal{T}x \in \mathcal{C}(\mathcal{M})$ and there exists $\mathcal{R} \in (0, \infty)$ such that $\mathcal{H}_q(\mathcal{T}fx, f\mathcal{T}x) \leq \mathcal{R}\ dist(fx, \mathcal{T}_\lambda x)$ for each $\lambda \in [0, 1]$, where $\mathcal{T}_\lambda x = (1-\lambda)p + \lambda\mathcal{T}x$. It is well known that $\mathcal{R}$-subweakly commuting maps are $\mathcal{R}$-weakly commuting(see [14]).

A set $\mathcal{M}$ is said to have property $(\mathcal{N})$ [9], if

(1) $\mathcal{T} : \mathcal{M} \to \mathcal{C}(\mathcal{M})$,

(2) $(1 - k_n)p + k_nTx \subseteq \mathcal{M}$,

for some $p \in \mathcal{M}$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1 and for each $x \in \mathcal{M}$. Each p-starshaped set has the property $(\mathcal{N})$ with respect to any map $\mathcal{T} : \mathcal{M} \to \mathcal{C}(\mathcal{M})$ but the converse does not hold, in general.

Example 1.1. *[1]. We will show that the q-th power of the quasi-norm $\|f\|_q$ in $\mathcal{L}_q$ defined by (1.1) is a q-norm on $\mathcal{L}_q$. For each $f \in \mathcal{L}_q$ the q-th power of the quasi-norm in $\mathcal{L}_q$ is defined by*

$$\|f\|_q^q = \int_a^b |f(t)|^q dt. \tag{1.3}$$

The norm defined by (1.3) is a q-norm on $\mathcal{L}_q$.

(1) For each $f \in \mathcal{L}_q$, $\|f\|_q \geq 0$. If $\|f\|_q^q = 0$ then $f(t) = 0$ almost everywhere,

(2) $\|af\|_q^q = \int_a^b |af(t)|^q dt = |a|^q \int_a^b |f(t)|^q dt = |a|^q \|f\|_q^q$ for all scalars a and all $f \in \mathcal{L}_q$,

(3) $\|f+g\|_q^q = \int_a^b |f(t)+g(t)|^q dt \leq \int_a^b |f(t)|^q dt + \int_a^b |g(t)|^q dt \leq \|f\|_q^q + \|g\|_q^q$ *for all* $f, g \in L_q$.

Thus all the properties q-norm, $0 < q \leq 1$, are satisfied. Hence the q-th power of the quasi-norm q in $\mathcal{L}_q$ is a q-norm on $\mathcal{L}_q$.

Fixed-point theorems have been used at many places in approximation theory. One of them is existence of best approximants where it is applied. Number of results exists in the literature applying the fixed-point theorem to prove the existence of best approximant.

Meinardus [8] was the first who employed a fixed-point theorem to establish the existence of an invariant approximation. Afterwards, Brosowski [2] obtained a celebrated result and generalized the Meinardus's result. Further, Singh [15] observed that the linearity of mapping and the convexity of the set of best approximants in the result of Brosowski [2], can be relaxed. In a subsequent paper, Singh [16] also observed that only the nonexpansiveness of mapping on set of best approximations is necessary for the validity of his own earlier result [15]. Later, Hicks and Humpheries [3] have shown that the result of Singh [15] remains true, if domain of mapping is replaced by the boundary of domain. Furthermore, Sahab, Khan and Sessa [13] generalized the result of Hicks and Humpheries [3] and Singh [15] by using a pair of commuting mappings, one linear and the other nonexpansive mapping and established the existence of best approximation in a normed space.

On the other hand, Latif and Tweddle [7] have proved some coincidence and common fixed point theorems for commuting pair of single-valued and multivalued mappings defined on starshaped subsets of a Banach space. Recently, Bano, Khan and Latif [1] extended the results of Lami Dezo [6] and Latif and Tweddle [7] in the setting of starshaped domain of q-normed space and used the result to establish invariant approximation result. Rhoades [11] and Shahzad [14] also extended the result of Latif and Tweddle [7] for noncommuting pair of maps in the setting of starshaped domain of Banach space. Shahzad [14] considers maps f and $\mathcal{T}$ satisfying a general condition $\mathcal{T}(\mathcal{M}) \subset f(\mathcal{M})$ while Rhoades [11] assume $f(\mathcal{M}) = \mathcal{M}$.

More recently, Hussain [4] improves the result of Rhoades [11] without the assumptions of $\mathcal{R}$-subweak commutativity of the pair $\{f, \mathcal{T}\}$, the continuity and affineness of the map f, completeness of the space $\mathcal{X}$ and replace the starshapedness of the set $\mathcal{M}$ by a weaker set of conditions by using result of Pathak and Khan [10].

The purpose of this paper is to show the validity of results of Hussain [4] in the framework of q-normed space which is not necessarily locally convex. As application of the result, an invariant approximation result is also obtained. Incidently, the results of Bano, Khan and Latif [1], Hussain [4], Latif and Tweddle [7], Rhoades [11], Sahab, Khan and Sessa [13], Shahzad [14] and Singh [15] are also extended and improved.

The following result is also needed in the sequel, which is a consequence of Theorem 3 of Pathak and Khan [10](see also Hussain [4, Theorem 2.1]).

Theorem 1.2. *Let* $(\mathcal{X}, d)$ *be a metric space,* $f : \mathcal{X} \to \mathcal{X}$ *and* $\mathcal{T} : \mathcal{X} \to \mathcal{C}(\mathcal{X})$ *such that* $\mathcal{T}(\mathcal{X}) \subset f(\mathcal{X})$*. If* $\mathcal{T}(\mathcal{X})$ *or* $f(\mathcal{X})$ *is complete and* $\mathcal{T}$ *is an* f*-contraction, then* $\mathcal{C}(f, \mathcal{T}) \neq \phi$*.*

2 Coincidence point theorem

We shall assume that $\mathcal{X}^*$ separates points of a q-normed space $\mathcal{X}$ whenever the weak topology is considered.

We begin with the following `coincidence point theorem` in the setting of q-normed space which improve and extend the Theorem 2.2 of Hussain [4].

Theorem 2.1. *Let* $\mathcal{M}$ *be a nonempty subset of a complete* q*-normed space* $\mathcal{X}$ *and* f *be a self-map on* $\mathcal{M}$*. Assume that* $\mathcal{T} : \mathcal{M} \to \mathcal{C}(\mathcal{M})$ *is* f*-nonexpansive map such that* $\mathcal{T}(\mathcal{M})$ *is bounded,* $f(\mathcal{M}) = \mathcal{M}$ *and* $(f - \mathcal{T})(\mathcal{M})$ *is closed. If* $\mathcal{M}$ *is complete and has the property* $(\mathcal{N})$*, then* $\mathcal{C}(f, \mathcal{T}) \neq \phi$*.*

Proof. Define $\mathcal{T}_n$ by $\mathcal{T}_n x = (1 - k_n)p + k_n \mathcal{T} x$ for all $x \in \mathcal{M}$ and fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1. Then, each $\mathcal{T}_n$ is a well-defined self-mapping of $\mathcal{M}$ as $\mathcal{M}$ has property $(\mathcal{N})$ and for each n, $\mathcal{T}_n(\mathcal{M}) \subset \mathcal{M} = f(\mathcal{M})$. Moreover, from the f-nonexpansiveness of $\mathcal{T}$, we have

$$\begin{aligned} \mathcal{H}_q(\mathcal{T}_n(x), \mathcal{T}_n(y)) &= (k_n)^q \mathcal{H}_q(\mathcal{T}(x), \mathcal{T}(y)) \\ &\leq (k_n)^q \|f(x) - f(y)\|_q \end{aligned}$$

i.e.,

$$\mathcal{H}_q(\mathcal{T}_n(x), \mathcal{T}_n(y)) \leq (k_n)^q \|f(x) - f(y)\|_q,$$

for each $x, y \in \mathcal{M}$ and $0 < k_n < 1$ and so $\mathcal{T}_n$ is an f-contraction. Thus, Theorem 1.2 guarantees that there exists an $x_n \in \mathcal{M}$ such that $x_n = f(x_n) \in$

$\mathcal{T}_n(x_n)$. So by the definition of $\mathcal{T}_n(x_n)$, there is a $y_n \in \mathcal{T}(x_n)$ such that

$$fx_n = (1-k_n)p + k_n y_n$$

$$fx_n - y_n = (1-k_n)p - (1-k_n)y_n$$

$$= (1-k_n)(p - y_n)$$

$$= (\tfrac{1}{k_n} - 1)(p - fx_n)$$

and

$$\|fx_n - y_n\|_q = (\frac{1}{k_n} - 1)^q \|p - fx_n\|_q \to 0 \ \ as \ \ n \to \infty$$

Since $\mathcal{T}(\mathcal{M}) \subseteq \mathcal{M}$ is bounded and $f(x_n) \in \mathcal{T}_n(x_n) \subseteq \mathcal{M}$, we have $\|f(x_n)\|_q$ is bounded so by the fact that $k_n \to 1$, we have $\|f(x_n) - y_n\|_q \to 0$. Now, as $fx_n - y_n \in (f - \mathcal{T})x_n$ and $(f - \mathcal{T})(\mathcal{M})$ is closed, we conclude that $0 \in (f - \mathcal{T})(\mathcal{M})$ and hence $\mathcal{C}(f, \mathcal{T}) \neq \phi$. This completes the proof. □

Remark 2.2. *Theorem 2.1 extends and improves Theorem* 2.2 *of Bano, Khan and Latif [1] without the assumptions of commutativity of the pair* $\{f, \mathcal{T}\}$*, the continuity and affineness of the map* f*, compactness of the subset* $\mathcal{M}$ *and replace the starshapedness of the set* $\mathcal{M}$ *by a weaker set of conditions and hence the result of Latif and Tweddle [7] (see also [11, Theorem 1]) in q-normed spaces.*

Remark 2.3. *Theorem 2.1 extends and improves Theorem* 3 *of Rhoades [11] without the assumptions of* $\mathcal{R}$*-subcommuting of the pair* $\{f, \mathcal{T}\}$*, the continuity and affineness of the map* f*, closedness of the subset* $\mathcal{M}$ *and replace the starshapedness of the set* $\mathcal{M}$ *by a weaker set of conditions.*

Theorem 2.4. *Let* $\mathcal{M}$ *be a nonempty weakly compact subset of a complete q-normed space* $\mathcal{X}$ *and* f *be a self-map on* $\mathcal{M}$*. Assume that* $\mathcal{T} : \mathcal{M} \to \mathcal{C}(\mathcal{M})$ *is* f*-nonexpansive map such that* $f(\mathcal{M}) = \mathcal{M}$ *and* $(f - \mathcal{T})$ *is demiclosed at* 0*. If* $\mathcal{M}$ *is complete and has the property* $(\mathcal{N})$*, then* $\mathcal{C}(f, \mathcal{T}) \neq \phi$*.*

Proof. As $\mathcal{T}(\mathcal{M}) \subset \mathcal{M}$ and $\mathcal{M}$ is bounded so $\mathcal{T}(\mathcal{M})$ is bounded. Hence, as in the proof of Theorem 2.1, $fx_n - y_n \to 0$ as $n \to \infty$ where $y_n \in \mathcal{T}x_n$. By the weak compactness of $\mathcal{M}$, there is a subsequence $\{x_m\}$ of the sequence $\{x_n\}$ such that $\{x_m\}$ converges weakly to $y \in \mathcal{M}$ as $m \to \infty$. Since $fx_m - y_m \in (f - \mathcal{T})(x_m)$ for each m and $(f - \mathcal{T})$ is demiclosed at 0, we obtain $0 \in (f - \mathcal{T})y$. Thus $\mathcal{C}(f - \mathcal{T}) \neq \phi$. □

Theorem 2.5. *Let $\mathcal{M}$ be a nonempty weakly compact subset of a complete q-normed space $\mathcal{X}$ satisfying Opial's condition and f be a weakly continuous self-map on $\mathcal{M}$. Assume that $\mathcal{T} : \mathcal{M} \to \mathcal{K}(\mathcal{M})$ is f-nonexpansive map such that $f(\mathcal{M}) = \mathcal{M}$. If $\mathcal{M}$ has the property $(\mathcal{N})$, then $\mathcal{C}(f, \mathcal{T}) \neq \phi$.*

Proof. By [1, Lemma 2.1], $(f - \mathcal{T})$ is demiclosed at 0. Hence the result follows from Theorem 2.4. □

Theorem 2.6. *Suppose that $\mathcal{X}$, $\mathcal{M}$, f, $\mathcal{T}$ and p satisfy the assumptions of Theorem 2.1. Moreover if f is continuous, $y \in \mathcal{C}(f, \mathcal{T})$ implies the existence of $\lim f^n y$ and the pair $\{f, \mathcal{T}\}$ is $\mathcal{R}$-weakly commuting (or $\mathcal{R}$-subweakly commuting), then $\mathcal{F}(f) \cap \mathcal{F}(\mathcal{T}) \neq \phi$.*

Proof. By Theorem 2.1, $\mathcal{C}(f, \mathcal{T}) \neq \phi$. If the pair $\{f, \mathcal{T}\}$ is $\mathcal{R}$-weakly commuting (or $\mathcal{R}$-subweakly commuting), then for any $y \in \mathcal{C}(f, \mathcal{T})$,

$$\mathcal{H}_q(f\mathcal{T}y, \mathcal{T}fy) \leq \mathcal{R}\|fy - \mathcal{T}y\|_q = 0.$$

This implies that $\{f, \mathcal{T}\}$ commutes on the points of $\mathcal{C}(f, \mathcal{T})$. It follows that $f^n y = f^{n-1} fy \in f^{n-1}\mathcal{T}y = \mathcal{T}f^{n-1}y$ for $y \in \mathcal{C}(f, \mathcal{T})$. If $z = lim_{z-1} f^n y$, taking $\lim_{n\to\infty}$, we get $z \in \mathcal{F}(\mathcal{T})$. Also $z \in \mathcal{F}(f)$. Hence $\mathcal{F}(f) \cap \mathcal{F}(\mathcal{T}) \neq \phi$. □

Theorem 2.7. *Suppose that $\mathcal{X}$, $\mathcal{M}$, f, $\mathcal{T}$ and p satisfy the assumptions of Theorem 2.4(or Theorem 2.5). Moreover if f is continuous, $y \in \mathcal{C}(f, \mathcal{T})$ implies the existence of $\lim f^n y$ and the pair $\{f, \mathcal{T}\}$ is $\mathcal{R}$-weakly commuting (or $\mathcal{R}$-subweakly commuting), then $\mathcal{F}(f) \cap \mathcal{F}(\mathcal{T}) \neq \phi$.*

Proof. The proof follows on the line of Theorem 2.6. □

An immediate consequence of the Theorem 2.1 is as follows:

Corollary 2.8. *Let $\mathcal{X}$ be a complete q-normed space and $\mathcal{M}$ be a complete subset of $\mathcal{X}$ which is starshaped with respect to $p \in \mathcal{M}$. Let $f : \mathcal{M} \to \mathcal{M}$ be such that $f(\mathcal{M}) = \mathcal{M}$ and let $\mathcal{T} : \mathcal{M} \to \mathcal{C}(\mathcal{M})$ be a f-nonexpansive map. If $(f - \mathcal{T})(\mathcal{M})$ is closed and $\mathcal{T}(\mathcal{M})$ is bounded, then $\mathcal{C}(f \cap \mathcal{T}) \neq \phi$.*

3 Invariant approximation

As an application of Theorem 2.1, we have the following result on `invariant approximation` which improve and extend the result of Bano, Khan and Latif [1, Theprem 2.4 and Corollary 2.5], Sahab, Khan and Sessa [13] and Singh [15]:

Theorem 3.1. *Let $\mathcal{M}$ be a subset of a complete q-normed space $\mathcal{X}$ and let $f : \mathcal{X} \to \mathcal{X}$ and $\mathcal{T} : \mathcal{X} \to \mathcal{C}(\mathcal{X})$ be mappings such that $u = fu$ and $\mathcal{T}u = \{u\}$ for some $u \in \mathcal{X}$ and $\mathcal{T}(\partial\mathcal{M} \cap \mathcal{M}) \subset \mathcal{M}$. Suppose that $f(\mathcal{P}_{\mathcal{M}}(u)) = \mathcal{P}_{\mathcal{M}}(u)$, and $\mathcal{T}$ is f-nonexpansive for all $x \in \mathcal{P}_{\mathcal{M}}(u) \cup \{u\}$. Then $\mathcal{P}_{\mathcal{M}}(u)$ is $\mathcal{T}$-invariant. Further, assume that $\mathcal{P}_{\mathcal{M}}(u)$ is complete, $\mathcal{T}(\mathcal{P}_{\mathcal{M}}(u))$ is bounded, $(f - \mathcal{T})(\mathcal{P}_{\mathcal{M}}(u))$ is closed and $\mathcal{P}_{\mathcal{M}}(u)$ has the property $(\mathcal{N})$, then $\mathcal{P}_{\mathcal{M}}(u) \cap \mathcal{C}(\mathcal{T}, f) \neq \phi$.*

Proof. Let $x \in \mathcal{P}_{\mathcal{M}}(u)$. Then $fx \in \mathcal{P}_{\mathcal{M}}(u)$ since $f(\mathcal{P}_{\mathcal{M}}(u)) = \mathcal{P}_{\mathcal{M}}(u)$. By the definition of $\mathcal{P}_{\mathcal{M}}(u)$, $x \in \partial\mathcal{M} \cap \mathcal{M}$ and since $\mathcal{T}(\partial\mathcal{M} \cap \mathcal{M}) \subset \mathcal{M}$, it follows that $\mathcal{T}x \subset \mathcal{M}$. Let $z \in \mathcal{T}(x)$. Then, by f-nonexpansiveness of $\mathcal{T}$,

$$\|z - u\|_q = \mathcal{H}_q(\mathcal{T}x, \mathcal{T}u) \leq \|fx - fu\|_q = \|fx - u\|_q = d_q(u, \mathcal{M}).$$

Now $z \in \mathcal{M}$ and $f(x) \in \mathcal{P}_{\mathcal{M}}(u)$ imply that $z \in \mathcal{P}_{\mathcal{M}}(u)$. Thus $\mathcal{T}(x) \in \mathcal{P}_{\mathcal{M}}(u)$. Hence $\mathcal{T}$ maps $\mathcal{P}_{\mathcal{M}}(u)$ into $\mathcal{C}(\mathcal{P}_{\mathcal{M}}(u))$. Now Theorem 2.1 guarantees that $\mathcal{P}_{\mathcal{M}}(u) \cap \mathcal{C}(\mathcal{T}, f) \neq \phi$. This completes the proof. □

References

[1] A. Bano, A. R. Khan and A. Latif, *Coincidence points and best approximations in p-normed spaces,* Rad. Math., 12(2003), 27 - 36.

[2] B. Brosowski, *Fixpunktsätze in der Approximationstheorie,* Mathematica(Cluj), 11(1969), 165 - 220.

[3] T. L. Hicks and M. D. Humpheries, *A note on fixed point theorems,* J. Approx. Theory, 34(1982), 221 - 225.

[4] N. Hussain, *Coincidence points for multivaluedmaps defined on non-starshaped domain,* Demonstratio Mathematica (to appear).

[5] G. Köthe, *Topological vector spaces I,* Springer-Verlag, Berlin, 1969.

[6] E. Lami Dozo, *Multivalued nonexpansive mappings and Opial's condition,* Proc. Amer. Math. Soc, 38 (1973), 286 - 292.

[7] A. Latif and I. Tweddle, *On multivalued f-nonexpansive maps,* Demonstratio Math, 32 (3)(1999), 565 - 574.

[8] G. Meinardus, *Invarianze bei Linearen Approximationen,* Arch. Rational Mech. Anal., 14(1963), 301 - 303.

[9] S. A. Naimpally, K. L. Singh, and J. H. M. Whitfield, *Fixed points and nonexpansive retracts in locally convex spaces,* Fundamenta Mathematicae, 120(1)(1984), 63 - 75.

[10] H. K. Pathak and M. S. Khan, *Fixed and coincidence points of hybrid mappings,* Arch. Math. (Brno), 38 (2002), 201-208.

[11] B. E. Rhoades, *On multivalued f-nonexpansive maps,* Fixed Point Theory and Appl., 2(2001), 89 - 92.

[12] W. Rudin, *Functional Analysis,* 2nd ed., International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1991.

[13] S. A. Sahab, M. S. Khan and S. Sessa, *A result in best approximation theory,* J. Approx. Theory, 55(1988), 349 - 351.

[14] N. Shahzad, *Coincidence points and $\mathcal{R}$-subweakly commuting multivalued maps,* Demonstratio Mathematica, 34(2003), 427 - 431.

[15] S. P. Singh, *An application of a fixed point theorem to approximation theory,* J. Approx. Theory, 25(1979), 89 - 90.

[16] S. P. Singh, *Application of fixed point theorems to approximation theory, in: V. Lakshmikantam(Ed.),* Applied nonlinear Analysis, Academic Press, New York, 1979.

Department of Mathematics, Raipur Institute of Technology
Chhatauna, Mandir Hasaud, Raipur-492101(Chhattisgarh), India
E-mail: hemantnashine@rediffmail.com
nashine_09@rediffmail.com