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# WARPED PRODUCT CONTACT CR-SUBMANIFOLDS OF TRANS-SASAKIAN MANIFOLDS

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**Abstract**

B.Y. Chen [4] showed that there exists no proper warped CR-submanifolds $N_\perp \times_f N_T$ of a Kaehler manifold and obtained many results on CR-warped products $N_T \times_f N_\perp$. Contact CR-warped product submanifolds in Sasakian manifold were studied by I. Hasegawa and I. Mihai [6]. In this paper we have investigated the existence of contact CR-warped product submanifolds in more general setting of trans-Sasakian manifolds.

## 1 Introduction

The study of warped product manifolds was initiated by R.L. Bishop and B. O'Neill [1] with differential geometric point of view. However the study got impetus only after B.Y. Chen's work on warped product CR-submanifolds of Kaehler manifold (cf., [4], [5]). Afterwards several papers appeared which have dealt with various geometric aspects of warped product submanifolds. Our aim in the paper is to study the warped product submanifolds in the setting of trans-Sasakian manifold which is more general to Sasakian one.

A $(2m+1)$-dimensional Riemannian manifolds $(\bar{M}, g)$ is said to be a *trans-Sasakian manifold* if it admits an endomorphism $\phi$ of its tangent bundle $T\bar{M}$, a vector field $\xi$ and a 1-form $\eta$ satisfying:

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$$\left.\begin{aligned}&\phi^2X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \phi(\xi) = 0,\ \eta \circ \phi = 0,\\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ \eta(X) = g(X, \xi),\\ &(\bar{\nabla}_X\phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X),\\ &\bar{\nabla}_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi)\end{aligned}\right\} \quad (1.1)$$

for any vector fields $X,\ Y$ on $\bar{M}$, where $\alpha$ and $\beta$ are smooth functions on $\bar{M}$ and $\bar{\nabla}$ denotes the Riemannian connection with respect to the Riemannian metric $g$. If $\alpha$ (respectively $\beta$) is zero then $\bar{M}$ is called $\beta-$*Kenmotsu* (respectively $\alpha-$*Sasakian*). If $\alpha$ and $\beta$ both are zero then the manifold $\bar{M}$ is called *Cosymplectic.*

Now, let $M$ be a submanifold immersed in $\bar{M}$. We also denote by $g$ the induced metric on $M$. Let $TM$ be the Lie algebra of vector fields in $M$ and $T^{\perp}M$ the set of all vector fields normal to $M$. Denote by $\nabla$ the Levi-Civita connection on $M$. Then Gauss and Weingarten formulas are given by

$$\bar{\nabla}_XY = \nabla_XY + h(X, Y) \quad (1.2)$$

and

$$\bar{\nabla}_XN = -A_NX + \nabla^{\perp}_XN \quad (1.3)$$

for any $X\ Y \in TM$ and any $N \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection in the normal bundle, $h$ is the second fundamental form of $M$ and $A_N$ is the Weingarten endomorphism associated with $N$. The second fundamental form $h$ and the shape operator $A$ are related by

$$g(A_NX, Y) = g(h(X, Y), N). \quad (1.4)$$

For any $X \in TM$, we write

$$\phi X = PX + FX \quad (1.5)$$

where $PX$ is the tangential component of $\phi X$ and $FX$ is the normal component of $\phi X$ respectively. Similarly, for any vector field $N$, normal to $M$, we put

$$\phi N = tN + fN \quad (1.6)$$

where $tN$ and $fN$ are tangential and normal components of $\phi N$, respectively.

The submanifold $M$ is said to be *invariant* if $F$ is identically zero. On the other hand, $M$ is said to be *anti-invariant* submanifold if $P$ is identically zero.

For submanifolds tangent to the structure vector field $\xi$, there are different classes of submanifolds. We mention the following

(i) A submanifold $M$ tangent to $\xi$ is called an *invariant* submanifold if $\phi$ preserves any tangent space of $M$, i.e., $\phi(T_pM) \subseteq T_PM$, for every $p \in M$.

(ii) A submanifold $M$ tangent to $\xi$ is called an *anti-invariant* submanifold if $\phi$ maps any tangent space of $M$ into the normal space, that is, $\phi(T_pM) \subset T_P^{\perp}M$, for every $p \in M$.

(iii) A submanifolds $M$ tangent to $\xi$ is called *contact CR-submanifold* if it admits an invariant distribution $\mathcal{D}$ whose orthogonal complementary distribution $\mathcal{D}^{\perp}$ is anti-invariant i.e., $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle\xi\rangle$ with $\phi(\mathcal{D}_p) \subseteq \mathcal{D}_p$ and $\phi(\mathcal{D}_p^{\perp}) \subset T_p^{\perp}M$, for every $p \in M$.

## 2 Warped product submanifolds

B.Y. Chen studied warped product CR-submanifolds in Kaehler manifolds and introduced the notion of CR-warped product [4]. Later I. Hasegawa and I. Mihai studied contact CR-warped product submanifolds in Sasakian manifolds [6]. In this paper we study warped product contact CR-submanifolds of trans-Sasakian manifolds which is more general than [6].

**Definition 2.1.** *Let $(B, g_1)$ and $(F, g_2)$ be two Riemannian manifolds with Riemannian metric $g_1$ and $g_2$ respectively and $f$ a positive differentiable function on $B$. The warped product of $B$ and $F$ is the Riemannian manifold $B \times_f F = (B \times F, g)$, where*

$$g = g_1 + f^2 g_2. \tag{2.1}$$

*More explicitly, if $U$ is tangent to $M = B \times_f F$ at $(p, q)$, then*

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p)\|d\pi_2 U\|^2$$

*where $\pi_i (i = 1,\ 2)$ are the canonical projections of $B \times F$ onto $B$ and $F$, respectively.*

We recall that on a warped product one has

$$\nabla_U V = \nabla_V U = (U \ln f)V \tag{2.2}$$

for any vector fields $U$ tangent to $B$ and $V$ tangent to $F$ [3].

If the manifolds $N_T$ and $N_\perp$ are invariant and anti-invariant submanifolds respectively of a trans-Sasakian manifold $\bar{M}$, then their warped products are
$(a)$ $N_\perp \times_f N_T$,
$(b)$ $N_T \times_f N_\perp$.

**Note.** In the sequel, we call the warped product submanifold $(a)$ as warped product CR-submanifold and the warped product $(b)$ as CR-warped product submanifold.

## 3 Warped product CR-submanifolds of a trans-Sasakian manifold

Throughout this section, we assume that $\bar{M}$ is a trans-Sasakian manifolds and $M = N_\perp \times_f N_T$ be a warped product CR-submanifold of trans-Sasakian manifold $\bar{M}$. Such submanifolds are always tangent to the structure vector field $\xi$. We distinguish 2 cases

(i) $\xi$ tangent to $N_T$,

(ii) $\xi$ tangent to $N_\perp$.

First we consider $\xi$ tangent to $N_T$.

**Theorem 3.1.** *Let $\bar{M}$ be a $(2m+1)$-dimensional trans-Sasakian manifold. Then there do not exist warped product CR-submanifold $M = N_\perp \times_f N_T$ such that $N_T$ is an invariant submanifold tangent to $\xi$ and $N_\perp$ is an anti-invariant submanifold of $\bar{M}$.*

*Proof.* Assume $M = N_\perp \times_f N_T$ is a warped product CR-submanifold of a trans-Sasakian manifold, such that $N_T$ is an invariant submanifold tangent to $\xi$ and $N_\perp$ is an anti-invariant submanifold of $\bar{M}$. By equation (2.2), we have

$$\nabla_X Z = \nabla_Z X = (Z \ln f)X \tag{3.1}$$

for any $X \in T(N_T)$ and $Z \in T(N_\perp)$.

In particular, for $X = \xi$, we have

$$\nabla_Z \xi = (Z \ln f)\xi. \tag{3.2}$$

Using the structure equation of trans-Sasakian manifold and equations (1.2) and (3.1), it follows that

$$-\alpha\phi Z + \beta(Z - \eta(Z)\xi) = \bar{\nabla}_Z \xi = \nabla_Z \xi + h(Z, \xi)$$

$$-\alpha F Z + \beta Z = \nabla_Z \xi + h(Z, \xi)$$

$$\Rightarrow \quad \left.\begin{aligned} \nabla_Z \xi &= \beta Z \\ h(Z, \xi) &= -\alpha F Z. \end{aligned}\right\} \tag{3.3}$$

From equations (3.2) and (3.3), we conclude that $Z \ln f = 0$, for all $Z \in T(N_\perp)$, i.e., $f$ is constant for all $Z \in T(N_\perp)$. This completes the proof of the theorem.

Now, we consider the case (ii)

**Theorem 3.2.** *Let $\bar{M}$ be a $(2m+1)$-dimensional trans-Sasakian manifold. Then there does not exist proper warped product CR-submanifold $M = N_\perp \times_f N_T$, such that $N_T$ is an invariant submanifold and $N_\perp$ is an anti-invariant submanifold tangent to $\xi$ of $\bar{M}$, unless $\bar{M}$ is $\beta-$ Kenmotsu.*

*Proof.* Assume $M = N_\perp \times_f N_T$ is a warped product submanifold of a trans-Sasakian manifold, such that $N_T$ is an invariant submanifold and $N_\perp$ is an anti-invariant submanifold tangent to $\xi$ of $\bar{M}$. In view of equation (2.2), we have

$$\nabla_X Z = \nabla_Z X = (Z \ln f)X \tag{3.4}$$

In particular, $Z = \xi$,

$$\nabla_X \xi = (\xi \ln f)X \tag{3.5}$$

From equations (1.1), (1.2) and (3.5), we get

$$\left.\begin{aligned} (\xi \ln f)X &= -\alpha\phi X + \beta X \\ h(X, \xi) &= 0. \end{aligned}\right\} \tag{3.6}$$

Since $X$ and $\phi X$ are linearly independent, from the above equation we see that proper warped product CR-submanifolds of $\bar{M}$ are possible only if $\alpha = 0$ and $\beta = \xi \ln f$ and in this case the ambient manifold becomes a Kenmotsu manifold, known as $\beta$-Kenmotsu. This completes the proof.

Now for the existence of warped product CR-submanifolds of type $N_\perp \times_f N_T$ in Kenmotsu manifold $\bar{M}$ of the type $N_\perp \times_f N_T$ , with $\xi$ tangent to $N_\perp$, we give the following example which shows that warped product CR-submanifolds of Kenmotsu manifolds exist in this case.

**Example.** Consider the complex space $C^4$ with the usual Kaehler structure and real global coordinates $(x^1, y^1, \cdots x^4, y^4)$. Let $\bar{M} = R \times \ _f C^4$ be the warped product between the real line $R$ and $C^4$, where the warping function is $f = e^t$, $t$ being the global coordinate on $R$. Then $\bar{M}$ is Kenmotsu manifold. Consider the distribution $D = span\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial y^2}\}$ and $D^\perp = span\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4}\}$ which obviously are integrable. Let us denote by $N_T$ and $N_\perp$ their, integral submanifolds, respectively. Let $g_{N_T} = \sum_{i=1}^{2}((dx^i)^2 + (dy^i)^2)$ and $g_{N_\perp} = dt^2 + e^{2t} \sum_{a=3}^{4} (dx^a)^2$ be Riemannian metrics on $N_T$ and $N_\perp$, respectively. Then, $M = N_\perp \times \ _f N_T$ is a contact CR-submanifold, isometrically immersed in $\bar{M}$. The warping function is given by $f = e^t$.

This example can also be generalized, replacing $C^4$ by $C^n$ and modifying the distributions $D$ and $D^\perp$ accordingly.

# 4 CR-warped product submanifolds of a trans-Sasakian manifold

In this section we study CR-warped product submanifolds $N_T \times_f N_\perp$. In this case also, we distinguish 2 cases

(i) $\xi$ tangent to $N_\perp$,

(ii) $\xi$ tangent to $N_T$.

We start with the case (i)

**Theorem 4.1.** *Let $\bar{M}$ be a $(2m+1)$-dimensional trans-Sasakian manifold. Then there does not exists CR-warped product submanifold $M = N_T \times_f N_\perp$ such that $N_\perp$ is an anti-invariant submanifold tangent to $\xi$ and $N_T$ is an invariant submanifold of $\bar{M}$.*

*Proof.* Let $M = N_T \times_f N_\perp$ is a CR-warped product submanifold of a trans-Sasakian manifold, such that $N_\perp$ is anti-invariant submanifold tangent to $\xi$ and $N_T$ is an invariant submanifold of $\bar{M}$. Let $X \in T(N_T)$ and $\xi \in T(N_\perp)$,then

$$\nabla_X \xi = (X \ln f)\xi. \tag{4.1}$$

Also, by last equation of (1.1) and equation (1.2), we have

$$-\alpha\phi X + \beta(X - \eta(X)\xi) = \bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi)$$

$$\left.\begin{aligned} &\nabla_X \xi = -\alpha\phi X + \beta X \\ &h(X, \xi) = 0. \end{aligned}\right\} \tag{4.2}$$

By equations (4.1) and (4.2), we get

$$(X \ln f)\xi = -\alpha\phi X + \beta X$$

$\Rightarrow X \ln f = 0$, for all $X \in T(N_T)$. This means that $f$ is constant for all $X \in T(N_T)$. This complete the proof of the theorem.

Now, we consieder the case (ii), i.e., $\xi$ tangent to $N_T$.

Assume $M = N_T \times_f N_\perp$ be a CR-warped product submanifold of trans-Sasakian manifold, such that $N_T$ is an invariant submanifold tangent to $\xi$ and $N_\perp$ is an anti-invariant submanifold of $\bar{M}$. Let $Z \in T(N_\perp)$ and $\xi \in T(N_T)$, we have

$$\nabla_Z \xi = (\xi \ln f)Z \tag{4.3}$$

Also, by structure equation and equation (1.2), we have

$$-\alpha\phi Z + \beta(Z - \eta(Z)\xi) = \bar{\nabla}_Z \xi = \nabla_Z \xi + h(Z, \xi)$$

$$\left.\begin{aligned} &\nabla_Z \xi = \beta Z \\ &h(X, \xi) = -\alpha F Z. \end{aligned}\right\} \tag{4.4}$$

From the equations (4.3) and (4.4), we get $(\xi \ln f) = \beta$, for all $Z \in T(N_\perp)$. In this case warped products do exist with smooth function $\beta \in C^\infty(N_T)$, which is a case similar to Theorem 3.2.

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