Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **21:2** (2007), 63–75

APPROXIMATING THE STIELTJES INTEGRAL VIA THE DARST-POLLARD INEQUALITY

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Abstract

An approximation of the Stieltjes integral of bounded integrals and continuous integrators via the Darst-Pollard inequality is given. Applications for the generalised trapezoid formula and the Ostrowski inequality for functions of bounded variation are also provided.

1 Introduction

In 1970, R. Darst and H. Pollard [2] obtained by elementary arguments the following interesting inequality for the Stieltjes integral.

Theorem 1 (Darst-Pollard, 1970). *If h is real and of bounded variation on the interval $[a,b]$ and g is real and continuous there, then*

$$\int_a^b h(t)\,dg(t) \leq \inf_{t\in[a,b]} h(t)\,[g(b)-g(a)] + S(g;a,b)\cdot \bigvee_a^b (h)\,, \tag{1.1}$$

where $\bigvee_a^b (h)$ is the total variation of h on $[a,b]$ and

$$S(g;a,b) := \sup_{a\leq\alpha<\beta\leq b} [g(\beta)-g(\alpha)]\,. \tag{1.2}$$

The Stieltjes integral $\int_a^b f(t)\,du(t)$ exists if f is continuous and u is of bounded variation, and $\int_a^b u(t)\,df(t)$ exists if and only if $\int_a^b f(t)\,du(t)$ exists.

2000 *Mathematics Subject Classification.* 26D15, 41A55.
Key words and phrases. Stieltjes integral, Darst-Pollard inequality, Generalised trapezoid rule, Ostrowski inequality.
Received: March 2, 2007

In the recent paper [4], in order to approximate the Stieltjes integral $\int_a^b f(t)\,du(t)$ by the quadrature rule

$$\frac{m+M}{2}\left[u(b)-u(a)\right],$$

where $m \le f(t) \le M$ for each $t \in [a,b]$, the first author defined the *error functional*

$$\Delta(f,u,m,M;a,b) := \int_a^b f(t)\,du(t) - \frac{m+M}{2}\left[u(b)-u(a)\right]$$

and showed that

$$|\Delta(f,u,m,M;a,b)| \le \begin{cases} \frac{1}{2}(M-m)\bigvee_a^b(u) & \text{if } u \text{ is of bounded variation,} \\ \frac{1}{2}(M-m)L(b-a) & \text{if } u \text{ is } L\text{-Lipschitzian,} \\ \int_a^b \left|f(t)-\frac{1}{2}(m+M)\right| du(t) & \text{if } u \text{ is monotonic nondecreasing.} \end{cases} \tag{1.3}$$

The constant $\frac{1}{2}$ is best possible in both inequalities. The last inequality in (1.3) is also sharp.

In the same paper [4], in order to approximate the Stieltjes integral $\int_a^b f(t)\,du(t)$ in terms of the *generalised trapezoid type* rule

$$\left[u(b)-\frac{n+N}{2}\right]f(b)+\left[\frac{n+N}{2}-u(a)\right]f(a),$$

the first author also considered the *error functional*

$$\nabla(f,u,n,N;a,b) := \left[u(b)-\frac{n+N}{2}\right]f(b)+\left[\frac{n+N}{2}-u(a)\right]f(a) - \int_a^b f(t)\,du(t),$$

where $-\infty < n \le u(t) \le N < \infty$ for $t \in [a,b]$ and showed that

$$|\nabla(f,u,n,N;a,b)| \le \begin{cases} \frac{1}{2}(N-n)\bigvee_a^b(f) & \text{if } f \text{ is of bounded variation,} \\ \frac{1}{2}(N-n)K(b-a) & \text{if } f \text{ is } K\text{-Lipschitzian,} \\ \int_a^b \left|u(t) - \frac{1}{2}(n+N)\right| df(t) & \text{if } f \text{ is monotonic nondecreasing.} \end{cases} \quad (1.4)$$

The constant $\frac{1}{2}$ is best possible in both inequalities above and the last one is sharp.

The main aim of the present paper is to provide other error estimates for the functionals Δ and ∇ defined above by utilising as a main tool the Darst-Pollard inequality stated in (1.1). Applications for the generalised trapezoid formula and the Ostrowski inequality for functions of bounded variation are also provided.

2 The Results

We can state the following result in estimating the error functional $\Delta(f,u,m,M;a,b)$:

Theorem 2. *Let u be continuous on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ of bounded variation such that*

$$-\infty < m = \inf_{t\in[a,b]} f(t), \sup_{t\in[a,b]} f(t) = M < \infty. \quad (2.1)$$

Then:

$$|\Delta(f,u,m,M;a,b)| \le \bigvee_a^b(f) \cdot S(u;a,b) - \frac{1}{2}(M-m)[u(b)-u(a)]. \quad (2.2)$$

The inequality (2.2) is sharp.

Proof. Let us denote $K(a,b) := \bigvee_a^b(f) \cdot S(u;a,b)$.

If we apply the Darst-Pollard inequality (1.1) for $h = f$ and $g = u$, we have

$$\int_a^b f(t)\,du(t) \le m[u(b)-u(a)] + K(a,b). \quad (2.3)$$

Now assume that $h := M - f$. Then h is of bounded variation on $[a,b]$,

$$\bigvee_a^b (h) = \bigvee_a^b (M - f) = \bigvee_a^b (f)$$

and $\inf_{t\in[a,b]} h(t) = 0$. On applying the Darst-Pollard inequality (1.1), we obtain

$$\int_a^b (M - f(t))\, du(t) \le K(a,b)$$

which is clearly equivalent with

$$M[u(b) - u(a)] - K(a,b) \le \int_a^b f(t)\, du(t). \tag{2.4}$$

Now, subtracting the same quantity $\frac{1}{2}(m+M)[u(b)-u(a)]$ in both (2.3) and (2.4), we deduce

$$\begin{aligned} &\frac{1}{2}(M-m)[u(b)-u(a)] - K(a,b) \qquad (2.5)\\ &\le \int_a^b f(t)\, du(t) - \frac{m+M}{2}[u(b)-u(a)] \\ &\le -\frac{1}{2}(M-m)[u(b)-u(a)] + K(a,b) \end{aligned}$$

which is clearly equivalent with the desired inequality (2.2).

For the sharpness of the inequality, let us assume that $u(t) = t$, $t \in [a,b]$. Then (1.2) becomes

$$\left|\int_a^b f(t)\, dt - \frac{m+M}{2}(b-a)\right| \le (b-a)\left[\bigvee_a^b (f) - \frac{1}{2}(M-m)\right]. \tag{2.6}$$

Now, if we choose the function $f : [a,b] \to \mathbb{R}$,

$$f(t) = \begin{cases} 0 & \text{if } t \in [a,b] \\ \\ k & \text{if } t = b,\ k > 0, \end{cases}$$

then we have $m = 0$, $M = k$, $\int_a^b f(t)\, dt = 0$, $\bigvee_a^b (f) = k$ and in both sides of (2.6) we obtain the same quantity $\frac{1}{2}k(b-a)$. □

The following particular cases are of interest.

Corollary 1. *Let $f:[a,b]\to\mathbb{R}$ be as in Theorem 2. If $u:[a,b]\to\mathbb{R}$ is of r-H-Hölder type (L-Lipschitzian), i.e.,*

$$|u(t)-u(s)|\le H|t-s|^r \quad (\le L|t-s|) \tag{2.7}$$

for any $t,s\in[a,b]$, where $H>0$, $r\in(0,1)$ $(L>0)$ are given, then:

$$|\Delta(f,u,m,M;a,b)|\le H(b-a)^r\bigvee_a^b(f)-\frac{1}{2}(M-m)[u(b)-u(a)] \tag{2.8}$$
$$\left(\le L(b-a)\bigvee_a^b(f)-\frac{1}{2}(M-m)[u(b)-u(a)]\right).$$

The case for Lipschitzian function u provides a sharp inequality.

The proof is obvious from the above theorem since

$$S(u;a,b)\le H(b-a)^r \quad (L(b-a))$$

and the sharpness of the inequality has been clearly proven for the function $u(t)=t$.

The case of absolutely continuous integrators $u:[a,b]\to\mathbb{R}$ is incorporated in the following corollary.

Corollary 2. *Let f be as in Theorem 2. If $u:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$, then*

$$|\Delta(f,u,m,M;a,b)| \le \begin{cases} (b-a)\|u'\|_{\infty,[a,b]}\bigvee_a^b(f)-\frac{1}{2}(M-m)[u(b)-u(a)] \\ \qquad \text{if} \quad u'\in L_\infty[a,b]; \\ \\ (b-a)^{\frac{1}{q}}\|u'\|_{p,[a,b]}\bigvee_a^b(f)-\frac{1}{2}(M-m)[u(b)-u(a)] \\ \qquad \text{if} \quad u'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \|u'\|_{1,[a,b]}\bigvee_a^b(f)-\frac{1}{2}(M-m)[u(b)-u(a)], \end{cases} \tag{2.9}$$

where $\|\cdot\|_{p,[a,b]}$ are the usual Lebesgue norms, $p\in[1,\infty]$.

Proof. Since u is absolutely continuous, hence for $a \leq \alpha < \beta \leq b$ we have

$$u(\beta) - u(\alpha) = \int_{\alpha}^{\beta} u'(s)\,ds \leq \begin{cases} (\beta-\alpha)\|u'\|_{\infty,[a,b]} & \text{if} \quad u' \in L_{\infty}[a,b]; \\ (\beta-\alpha)^{\frac{1}{q}}\|u'\|_{p,[a,b]} & \text{if} \quad u' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|u'\|_{1,[a,b]}; \end{cases}$$

which gives

$$S(u;a,b) \leq \begin{cases} (b-a)\|u'\|_{\infty,[a,b]} & \text{if} \quad u' \in L_{\infty}[a,b]; \\ (b-a)^{\frac{1}{q}}\|u'\|_{p,[a,b]} & \text{if} \quad u' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|u'\|_{1,[a,b]}. \end{cases}$$

These together with (2.2) produces (2.9). □

The case of monotonic integrators is considered in the following.

Corollary 3. *Let $f : [a,b] \to \mathbb{R}$ be as in Theorem 2 and $u : [a,b] \to \mathbb{R}$ a continuous and monotonic function on $[a,b]$. Then*

$$|\Delta(f,u,m,M;a,b)| \leq [u(b)-u(a)]\left[\bigvee_a^b(f) - \frac{1}{2}(M-m)\right] \tag{2.10}$$
$$\left(\leq \frac{1}{2}(M-m)[u(b)-u(a)]\right).$$

The inequality is sharp.

The proof is obvious from the fact that for monotonic functions $u : [a,b] \to \mathbb{R}$ one can derive that $S(u;a,b) = u(b) - u(a)$.

The following lemma may be stated (see also [4]).

Lemma 1. *Let $u : [a,b] \to \mathbb{R}$ and $\varphi, \phi \in \mathbb{R}$ with $\phi > \varphi$. The following statements are equivalent:*

(i) The function $u - \frac{\varphi+\Phi}{2} \cdot e$, where $e(t) = t$, $t \in [a,b]$ is $\frac{1}{2}(\Phi-\varphi)$-Lipschitzian;

(ii) We have the inequality:

$$\varphi \leq \frac{u(t) - u(s)}{t - s} \leq \Phi \quad \text{for each } t, s \in [a, b] \quad \text{with } t > s. \tag{2.11}$$

(iii) We have the inequality:

$$\varphi(t - s) \leq u(t) - u(s) \leq \Phi(t - s) \quad \text{for each } t, s \in [a, b] \quad \text{with } t > s. \tag{2.12}$$

Following [5], we can introduce the concept

Definition 1. *The function $u : [a, b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) is said to be (φ, Φ)-Lipschitzian on $[a, b]$.*

Notice that in [5], the definition was introduced on utilising the statement (iii) and only the equivalence (i) $\Longleftrightarrow$ (iii) was considered.

Utilising Lagrange's mean value theorem, we can state the following result that provides practical examples of (φ, Φ)-Lipschitzian functions:

Proposition 1. *Let $u : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $[a, b]$. If*

$$-\infty < \gamma = \inf_{t \in (a,b)} u'(t), \sup_{t \in (a,b)} u'(t) = \Gamma < \infty,$$

then u is (Γ, γ)-Lipschitzian on $[a, b]$.

Now the following corollary can be stated as well.

Corollary 4. *If $f : [a, b] \to \mathbb{R}$ is as in Theorem 2 and $u : [a, b] \to \mathbb{R}$ is (φ, Φ)-Lipschitzian with $\varphi > 0$, then*

$$|\Delta(f, u, m, M; a, b)| \leq \left[\Phi \cdot \bigvee_a^b (f) - \frac{1}{2}(M - m) \cdot \varphi\right](b - a). \tag{2.13}$$

The following result may be stated as well.

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $u : [a, b] \to \mathbb{R}$ a function of bounded variation such that*

$$-\infty < n = \inf_{t \in [a,b]} u(t), \sup_{t \in [a,b]} u(t) = N < \infty.$$

Then we have the inequality:

$$|\nabla(f, u, n, N; a, b)| \leq \bigvee_a^b (u) S(f; a, b) - \frac{1}{2}(N - n)[f(b) - f(a)]. \tag{2.14}$$

The inequality is sharp.

Proof. Follows from Theorem 2 on utilising the identity

$$
\begin{aligned}
& f(b)\left[u(b)-\frac{n+N}{2}\right]+f(a)\left[\frac{n+N}{2}-u(a)\right]-\int_a^b f(t)\,du(t) \\
&= \int_a^b \left[u(t)-\frac{n+N}{2}\right] df(t) \\
&= \int_a^b u(t)\,df(t)-\frac{n+N}{2}\left[f(b)-f(a)\right].
\end{aligned}
$$

The details are omitted. □

Similar corollaries to Corollary 1 – Corollary 4 may be stated. We leave them to the interested reader.

3 Applications to the Trapezoid Rule

In this section we provide some applications in connection with the generalised trapezoid rule.

In [1], in order to approximate the integral $\int_a^b f(t)\,dt$ for the function $f:[a,b]\to\mathbb{R}$ of bounded variation with the generalised trapezoid rule

$$f(a)(x-a)+f(b)(b-x), \qquad x\in[a,b]$$

the authors have considered the *generalised trapezoid error functional*

$$T(f;a,b,x):=\int_a^b f(t)\,dt-\left[f(a)(x-a)+f(b)(b-x)\right],$$

and obtained the following sharp bound

$$|T(f;a,b,x)|\le\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f) \tag{3.1}$$

for each $x\in[a,b]$.

The best inequality we can derive from (3.1) is the following trapezoid inequality for functions of bounded variation:

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}\cdot(b-a)\right|\le\frac{1}{2}(b-a)\bigvee_a^b(f), \tag{3.2}$$

where the constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller constant.

The version of (3.2) for continuous functions is incorporated in

Proposition 2. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then*

$$\left| \frac{f(a) + f(b)}{2} (b - a) - \int_a^b f(t)\, dt \right| \leq (b - a) \left\{ S(f; a, b) - \frac{1}{2} [f(b) - f(a)] \right\}, \quad (3.3)$$

where, as above

$$S(f; a, b) = \sup_{a \leq \alpha < \beta \leq b} [f(\beta) - f(\alpha)].$$

Proof. We use the identity [1]

$$f(b)(b - x) + f(a)(x - a) - \int_a^b f(t)\, dt = \int_a^b (t - x)\, df(t) \quad (3.4)$$

for any $x \in [a, b]$.

On applying Theorem 2 for the Stieltjes integral $\int_a^b (t - x)\, df(t)$ we can write that

$$\left| \int_a^b (t - x)\, df(t) - \frac{b - x + a - x}{2} [f(b) - f(a)] \right| \leq (b - a) S(f; a, b) - \frac{1}{2} (b - a) [f(b) - f(a)]. \quad (3.5)$$

Finally, on utilising the identity (3.4) and the inequality (3.5), a simple calculation provides the desired inequality (3.3). □

The following result may be stated as well.

Proposition 3. *Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous and such that the derivative f' is of bounded variation on $[a, b]$. If $-\infty < \gamma = \inf_{t \in [a,b]} f'(t)$, $\sup_{t \in [a,b]} f'(t) = \Gamma < \infty$, then*

$$\left| f(b)(b - x) + f(a)(x - a) + \frac{\gamma + \Gamma}{2} (b - a) \left(x - \frac{a + b}{2} \right) - \int_a^b f(t)\, dt \right| \leq (b - a) \left[\bigvee_a^b (f') \left[\frac{1}{2} (b - a) + \left| x - \frac{a + b}{2} \right| \right] - \frac{1}{2} (\Gamma - \gamma) \left(\frac{a + b}{2} - x \right) \right]. \quad (3.6)$$

Proof. If f is absolutely continuous on $[a,b]$, then

$$f(b)(b-x)+f(a)(x-a)-\int_a^b f(t)\,dt=\int_a^b f'(t)\,d\left[\frac{1}{2}(t-x)^2\right]$$

for any $x\in[a,b]$.

For fixed $x\in[a,b]$, let $u(t)=\frac{1}{2}(t-x)^2$. Then u is differentiable and $u'(t)=(t-x)$. Also

$$\begin{aligned}\sup_{t\in[a,b]}\left|u'(t)\right| &= \sup_{t\in[a,b]}|t-x|=\max\{x-a,b-x\}\\ &=\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|.\end{aligned}$$

Now, if we apply Corollary 1 for the Lipschitzian case, we can write:

$$\begin{aligned}&\left|\int_a^b f'(t)\,d\left[\frac{1}{2}(t-x)^2\right]-\frac{\gamma+\Gamma}{2}\left[\frac{(b-x)^2-(a-x)^2}{2}\right]\right|\\ &\qquad\le(b-a)\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f')\\ &\qquad\qquad-\frac{1}{2}(\Gamma-\gamma)\left[\frac{(b-x)^2-(a-x)^2}{2}\right],\end{aligned}$$

which is equivalent with the desired inequality (3.6). □

Corollary 5. *With the assumptions as in Proposition 3, we have the trapezoid inequality:*

$$\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\right|\le\frac{1}{2}(b-a)^2\bigvee_a^b(f'). \qquad (3.7)$$

4 Applications to the Ostrowski Inequality

In 1938, A. Ostrowski proved the following inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)M \qquad (4.1)$$

for all $x\in[a,b]$, provided that f is differentiable on (a,b) and $|f'(t)|\le M$ for all $t\in(a,b)$.

Using the following representation, which has been obtained by Montgomery in an equivalent form [6, p. 565]

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b p(t,x)\,f'(t)\,dt \tag{4.2}$$

for all $x \in [a,b]$, provided that f is absolutely continuous on $[a,b]$ and

$$p(t,x) := \begin{cases} t-a & \text{if} \quad t \in [a,x] \\ \\ t-b & \text{if} \quad t \in (x,b\} \end{cases}, (x,t) \in [a,b]^2, \tag{4.3}$$

we can put in place of M, i.e., in (4.1), the sup norm of f', i.e., $\|f'\|_\infty$ where

$$\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)|,$$

provided that $f' \in L_\infty[a,b]$.

The following result related to the Ostrowski inequality can be stated.

Proposition 4. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Then*

$$\left| f(x)(b-a) - \left(x - \frac{a+b}{2}\right)[f(b)-f(a)] - \int_a^b f(t)\,dt \right| \leq (b-a)\left[S(f;a,b) - \frac{1}{2}[f(b)-f(a)]\right], \tag{4.4}$$

where, as above, $S(f;a,b) = \sup_{a\leq\alpha<\beta\leq b}[f(\beta)-f(\alpha)]$.

Proof. We use the following Montgomery type identity [3]

$$\int_a^b p(t,x)\,df(t) = f(x)(b-a) - \int_a^b f(t)\,dt$$

for any $x \in [a,b]$, where the kernel $p : [a,b]^2 \to \mathbb{R}$ is defined by (4.3).

For any fixed $x \in [a,b]$, the function $p(\cdot,x)$ is of bounded variation, and

$$\bigvee_a^b p(\cdot,x) = \bigvee_a^x p(\cdot,x) + \bigvee_x^b p(\cdot,x) = x-a+b-x = b-a.$$

Also

$$\sup_{t\in[a,b]} p(t,x) = x-a \qquad \text{and} \qquad \inf_{t\in[a,b]} p(t,x) = x-b,$$

for any $x \in [a, b]$.

Now, if we apply Theorem 2 for the Stieltjes integral $\int_a^b p(t, x)\, df(t)$, then we can write:

$$\left| \int_a^b p(t, x)\, df(t) - \frac{1}{2}(x - a + x - b)[f(b) - f(a)] \right| \leq (b - a) S(f; a, b) - \frac{1}{2}(b - a)[f(b) - f(a)],$$

for any $x \in [a, b]$, which is equivalent with the desired result (4.4). □

Corollary 6. *With the assumptions as in Proposition 4 we have the mid-point inequality:*

$$\left| f\left(\frac{a+b}{2}\right)(b - a) - \int_a^b f(t)\, dt \right| \leq (b - a)\left[S(f; a, b) - \frac{1}{2}[f(b) - f(a)]\right]. \quad (4.5)$$

Remark 1. *The interested reader can apply Theorem 2 and Theorem 3 for other quadrature rules where the remainder can be written in the form of*

$$\int_a^b K_n(t, x)\, df^{(n)}(t)$$

where the n-th derivative of the integrand f is continuous and the Peano kernel $K_n(\cdot, x)$ is of bounded variation on $[a, b]$. The details are omitted.

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