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A GENERALISED COMMUTATIVITY THEOREM FOR PK-QUASIHYPONORMAL OPERATORS

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Abstract

For Hilbert space operators A and B, let δ_{AB} denote the generalised derivation $\delta_{AB}(X) = AX - XB$ and let $\triangle_{AB}$ denote the elementary operator $\triangle_{AB}(X) = AXB - X$. If A is a pk-quasihyponormal operator, $A \in pk - QH$, and B^* is an either p-hyponormal or injective dominant or injective $pk - QH$ operator (resp., B^* is an either p-hyponormal or dominant or $pk - QH$ operator), then $\delta_{AB}(X) = 0 \Longrightarrow \delta_{A^*B^*}(X) = 0$ (resp., $\triangle_{AB}(X) = 0 \Longrightarrow \triangle_{A^*B^*}(X) = 0$).

1 Introduction

Let $B(\mathcal{H},\mathcal{K})$, $B(\mathcal{H}) = B(\mathcal{H},\mathcal{H})$, denote the algebra of operators (equivalently, bounded linear transformations) from a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$. Let $\delta_{AB} \in B(B(\mathcal{K}), B(\mathcal{H}))$, $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$, denote the *generalised derivation* $\delta_{AB}(X) = AX - XB$, and let $\triangle_{AB} \in B(B(\mathcal{K}), B(\mathcal{H}))$ denote the *elementary operator* $\triangle_{AB}(X) = AXB - X$. The (classical) Putnam-Fuglede commutativity theorem says that if A and B are normal operators, then $\delta^{-1}_{AB}(0) \subseteq \delta^{-1}_{A^*B^*}(0)$. Over the years, the Putnam-Fuglede commutativity theorem has been extended to various classes of operators, each more general than the class of normal operators, and to the elementary operator $\triangle_{AB}$ to prove that $\triangle^{-1}_{AB}(0) \subseteq \triangle^{-1}_{A^*B^*}(0)$ for many of these classes of operators (see [1, 2, 3, 7] and [9] for references). Recall that an operator $A \in B(\mathcal{H})$ is said to be (p,k)-quasihyponormal, $A \in pk - QH$, for some real number $0 < p \leq 1$ and non-negative integer k (momentarily, we allow $k = 0$) if $A^{*k}(|A|^{2p} - |A^*|^{2p})A^k \geq 0$. Evidently, a $10 - QH$ operator is *hyponormal*,

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a $p0-QH$ operator is *p-hyponormal*, a $11-QH$ operator is *quasihyponormal* and a $1k-QH$ operator for $k \geq 1$ is *k-quasihyponormal*. Recently, Kim [9, Theorem 11] has proved that if $A \in B(\mathcal{H})$ is an injective $pk-QH$ operator, $k \geq 1$, and $B^* \in B(\mathcal{K})$ is a p-hyponormal operator, then $\delta_{AB}^{-1}(0) \subseteq \delta_{A^*B^*}^{-1}(0)$. Using what is essentially a very simple argument, we prove in this note a more general result which not only leads to Kim's result (*loc.cit.*) but also gives us further similar results. Thus we prove that if $A \in pk-QH$ and B^* is either p-hyponormal or injective dominant or injective $pk-QH$, then $\delta_{AB}^{-1}(0) \subseteq \delta_{A^*B^*}^{-1}(0)$, and if $A \in pk-QH$ and B^* is either p-hyponormal or dominant or $pk-QH$, then $\triangle_{AB}^{-1}(0) \subseteq \triangle_{A^*B^*}^{-1}(0)$. We also consider operators $A \in pk-QH$ for which $\delta_{A^*A}(X) = 0$ or $\triangle_{AA^*}(X) = 0$ for some invertible operator X.

In the following we shall denote the closure of a set S by $\overline{S}$, the range of $T \in B(\mathcal{H})$ by $T\mathcal{H}$ or by ran$T$, the orthogonal complement of $T^{-1}(0)$ by $\ker^{\perp}T$, the spectrum of T by $\sigma(T)$, the point spectrum of T by $\sigma_p(T)$, and the class of p-hyponormal operators, $0 < p < 1$, by $p-H$. Recall that an operator T is a quasiaffinity if it is injective and has dense range. Any other notation or terminology will be defined at the first instance of its occurrence.

2 Results

Let $\mathcal{P}_1$ denote the class of operators $A \in B(\mathcal{H})$ such that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix},$$

where $A_{22}^k = 0$ for some integer $k \geq 1$, and let $\mathcal{P}_2$ denote the class of operators $B \in B(\mathcal{K})$ such that B has the decomposition $B = B_n \oplus B_p$ into its normal and pure (= completely non-normal) parts, with respect to some decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, such that B_p has dense range. Let $(\mathcal{P}_1, \mathcal{P}_2)$ (resp., $[\mathcal{P}_1, \mathcal{P}_2]$) denote the class of operators (A, B), $A \in \mathcal{P}_1$ and $B \in \mathcal{P}_2$, such that $\delta_{A_{11}B_p}^{-1}(0) \subseteq \delta_{A_{11}^*B_p^*}^{-1}(0)$ and $\delta_{A_{11}B_n}^{-1}(0) \subseteq \delta_{A_{11}^*B_n^*}^{-1}(0)$ (resp., $\triangle_{A_{11}B_n}^{-1}(0) \subseteq \triangle_{A_{11}^*B_n^*}^{-1}(0)$.) The following theorem is our main result.

Theorem 2.1. *(i) If $(A, B) \in (\mathcal{P}_1, \mathcal{P}_2)$ and $X \in B(\mathcal{K}, \mathcal{H})$ is a quasiaffinity, then $X \in \delta_{AB}^{-1}(0) \Longrightarrow X \in \delta_{A^*B^*}^{-1}(0)$.*

(ii) If $(A, B) \in [\mathcal{P}_1, \mathcal{P}_2]$ and $X \in B(\mathcal{K}, \mathcal{H})$ is a quasiaffinity, then $X \in \triangle_{AB}^{-1}(0) \Longrightarrow X \in \triangle_{A^*B^*}^{-1}(0)$.

Proof. Let the quasiaffinity $X : \mathcal{K}_1 \oplus \mathcal{K}_2 \longrightarrow \mathcal{H}_1 \oplus \mathcal{H}_2$ have the matrix representation $X = [X_{ij}]_{i,j=1}^2$.

(i) If $X \in \delta_{AB}^{-1}(0)$, then $\delta_{A_{22}B_p}(X_{22}) = 0 \Longrightarrow X_{22}B_p^k = 0 \Longrightarrow X_{22} = 0$, since B_p has dense range. Evidently, the hypothesis $\delta_{A_{11}B_p}^{-1}(0) \subseteq \delta_{A_{11}^*B_p^*}^{-1}(0) \Longrightarrow \delta_{A_{11}B_p}(X_{12}) = 0 = \delta_{A_{11}^*B_p^*}(X_{12}) \Longrightarrow \overline{\text{ran}X_{12}}$ reduces A_{11}, $\ker^{\perp}X_{12}$ reduces B_p, and $A_{11}|_{\overline{\text{ran}X_{12}}}$ and $B_p|_{\ker^{\perp}X_{12}}$ are unitarily equivalent normal operators. Hence $X_{12} = 0$, which since X is a quasiaffinity implies that $\mathcal{K}_2 = \mathcal{H}_2 = \{0\}$. Consequently, the hypothesis $\delta_{A_{11}B_n}^{-1}(0) \subseteq \delta_{A_{11}^*B_n^*}^{-1}(0)$ implies that $\delta_{A^*B^*}(X) = 0$.

(ii) If $\triangle_{AB}(X) = 0$, then

$$\triangle_{A_{22}B_n}(X_{21}) = 0 \Longrightarrow X_{21} = A_{22}X_{21}B_n = A_{22}^k X_{21}B_n^k = 0,$$

and

$$\triangle_{A_{22}B_p}(X_{22}) = 0 \Longrightarrow X_{22} = A_{22}X_{22}B_p = A_{22}^k X_{22}B_p^k = 0.$$

Since X is a quasiaffinity, $\mathcal{K}_2 = \mathcal{H}_2 = \{0\}$. Consequently, the hypothesis $\triangle_{A_{11}B_n}^{-1}(0) \subseteq \triangle_{A_{11}^*B_n^*}^{-1}(0)$ implies that $\triangle_{A^*B^*}(X) = 0$. $\square$

The *numerical range* $W(T)$ of an operator $T \in B(\mathcal{H})$ is the set $\{\langle Tx, x\rangle : ||x|| = 1\}$. Recall from Embry [6] that if A and $B \in B(\mathcal{H})$ are commuting normal operators, and if $X \in B(\mathcal{H})$ is such that $0 \notin \overline{W(X)}$ and $\delta_{AB}(X) = 0$, then $A = B$. Thus, if A is a normal operator such that $\delta_{A^*A}(X) = 0$ for some operator X such that $0 \notin \overline{W(X)}$, then A is self-adjoint. That a similar result holds for operators $A \in p - H$ is proved in [9, Theorem 2]. In the following we prove an analogue of Embry's result for operators $A \in \mathcal{P}_1$ such that $\delta_{A^*A}(X) = 0$ or $\triangle_{A^*A}(X) = 0$. Let $\partial\mathbf{D}$ denote the boundary of the unit disc.

Theorem 2.2. *Let $A \in \mathcal{P}_1$ have the decomposition $A = A_n \oplus A_p$ into its normal and pure parts (alongwith the matrix decomposition above). Let $X \in B(\mathcal{H})$ be invertible.*

(i) If $0 \notin \overline{W(X)}$, $\delta_{A^*A}(X) = 0$ and $\delta_{A_{11}^*A_p}^{-1}(0) \subseteq \delta_{A_{11}A_p^*}^{-1}(0)$, then A is the direct sum of a self-adjoint operator and a nilpotent operator (where either component may act on the trivial space.)

(ii) If $\triangle_{A^*A}(X) = 0$ and $\triangle_{A_{11}^*A_p}^{-1}(0) \subseteq \triangle_{A_{11}A_p^*}^{-1}(0)$, then A is unitary.

Proof. (i) If we let A^* have the matrix representation above, $A = A_n \oplus A_p$ and $X = [X_{ij}]_{i,j=1}^2$, then $\delta_{A_{11}^*A_p}(X_{12}) = 0$. Hence, since $\delta_{A_{11}^*A_p}^{-1}(0) \subseteq \delta_{A_{11}A_p^*}^{-1}(0)$, $\ker^{\perp}(X_{12})$ reduces A_p, and $A_{11}^*|_{\overline{\text{ran}(X_{12})}}$ and $A_p|_{\ker^{\perp}X_{12}}$ are unitarily equivalent normal operators [3, Lemma 1(i)]. Since A_p is pure, we must have that $X_{12} = 0$. Consequently, X_{22} is injective. Since $X_{22} \in \delta^{-1}_{A^*_{22}A_p}(0)$ and ${A^*_{22}}^k = 0$, $X_{22}A^k_p = 0 \Longrightarrow A_p$ is k-nilpotent. To complete the proof we observe now that if $A^* = XAX^{-1}$ and $0 \notin \overline{W(X)}$, then $\sigma(A)$ is real [9, Lemma 3]. Since $\sigma(A) = \sigma(A_n) \cup \sigma(A_p)$ and A_n is normal, A_n is self-adjoint.

(ii) Representing A^*, A and X as in (i) above, it is seen that

$$\triangle_{A_{11^*}A_p}(X_{12}) = 0 = \triangle_{A_{11}A^*_p}(X_{12}),$$

and hence that $\ker^{\perp}(X_{12})$ reduces A_p and $A_p|\ker^{\perp}(X_{12})$ is normal [3, Lemma 1(ii)]. Since A_p is pure, $X_{12} = 0$ and $X_{22} \in \triangle^{-1}_{A^*_{22}A_p}(0)$. Since A_{22} is k-nilpotent, $X_{22} = 0 \Longrightarrow X = X_{11}$ and $A = A_n$ is normal. Hence $A^*XA = X = AXA^*$ [3, Corollary3] $\Longrightarrow |X|^2 A = (AX^*A^*)XA = (AX^*)A^*XA = A|X|^2 \Longrightarrow |X|A = A|X|$. Letting X have the polar decomposition $X = U|X|$, it follows that $A^*UA = U \Longrightarrow A$ is invertible and A^{-1} is unitarily equivalent to A^*. Hence $\sigma(A) \subseteq \partial\mathbf{D}$. Since A is normal, A is unitary. □

Remark 2.3. *(i)* Theorem 2.2(i) has a more satisfactory form for $pk - QH$ operators. Thus, if an operator $A \in pk - QH$ is such that $\delta_{A^*A}(X) = 0$ for some invertible operator X such that $0 \notin \overline{W(X)}$, then $\sigma(A)$ is real. Hence A_n in the decomposition $A = A_n \oplus A_p$ being normal is self-adjoint. Since a $pk - QH$ operator with zero Lebsgue area measure is the direct sum of a normal operator with a nilpotent operator [8, Corollary 6], A_p is nilpotent and A is the direct sum of a self-adjoint operator with a nilpotent operator. Hence: If $A \in pk - QH$, $0 \notin \overline{W(X)}$ and $\delta_{A^*A}(X) = 0$, then A is the direct sum of a self-adjoint operator with a nilpotent operator (*cf.* [9, Theorem 5]). Observe that if the operator A of Theorem 2.2(i) is reduction normaloid (i.e., the restriction of A to reducing subspaces of A is normaloid), then A is self-adjoint. Although $pk - QH$ operators are not normaloid, $p - H$ operators are (reduction normaloid). Hence, if $A \in p - H$, $0 \notin \overline{W(X)}$ and $\delta_{A^*A}(X) = 0$, then A is a self-adjoint operator [9, Theorem 2].
(ii) Let $A \in pk - QH$ and assume that $\triangle_{A^*A}(X) = 0$ for some invertible operator X. Then A is left invertible. Hence, if A has a dense range (evidently, see definition, such a $pk - QH$ operator is a $p - H$ operator), then A is invertible, and so $\sigma(A) \subseteq \partial\mathbf{D}$. Since a $p - H$ operator with spectrum in $\partial\mathbf{D}$ is unitary, A is unitary.

Applications. The restriction of a $pk - QH$ operator to an invariant subspace is again a $pk - QH$ operator. We assume in the following that $0 < p < 1$ and $k \geq 1$. Recall that every $A \in pk - QH \cap B(\mathcal{H})$ has a representation

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} \overline{T^k\mathcal{H}} \\ T^{*k^{-1}}(0) \end{pmatrix},$$

where $A_{11} \in p-H$ and $A_{22}^k = 0$ [8]. Evidently $\delta^{-1}_{A_{11}N}(0) \subseteq \delta^{-1}_{A^*_{11}N^*}(0)$ for every normal operator N and $\delta^{-1}_{A_{11}T}(0) = \{0\}$ for all pure p-hyponormal or dominant operators T^* [2, Theorem 7 and Corollary 8]. If S is a hyponormal operator and T^* is a dominant operator, then $\triangle^{-1}_{ST}(0) \subseteq \triangle^{-1}_{S^*T^*}(0)$: this is a consequence of the fact that $\delta^{-1}_{ST}(0) \subseteq \delta^{-1}_{S^*T^*}(0)$ (see [1, Theorem 1] and [3, Theorem 2]). Since to every p-hyponormal operator R there corresponds a hyponormal operator S and a quasiaffinity X such that $XR = RS$ [4, Proof of Theorem 1], it follows that $\triangle^{-1}_{A_{11}T}(0) \subseteq \triangle^{-1}_{A^*_{11}T^*}(0)$ for every p-hyponormal or dominant operator T^*.

Theorem 2.4. *Let $A \in pk-QH \cap B(\mathcal{H})$ and $B \in B(\mathcal{K})$. If B^* is an either p-hyponormal or injective dominant or injective $pk-QH$ operator (resp., B^* is an either p-hyponormal or dominant or $pk-QH$ operator), then $\delta^{-1}_{AB}(0) \subseteq \delta^{-1}_{A^*B^*}(0)$ (resp., $\triangle^{-1}_{AB}(0) \subseteq \triangle^{-1}_{A^*B^*}(0)$).*

Proof. Let d_{AB} stand for either of δ_{AB} and $\triangle_{AB}$. For a $Y \in d^{-1}_{AB}(0)$, define the quasiaffinity $X : \ker^{\perp}Y \longrightarrow \overline{\text{ran}Y}$ by setting $Xx = Yx$ for each $x \in \mathcal{K}$. Evidently, $\overline{\text{ran}Y}$ is invariant for A and $\ker^{\perp}Y$ is invariant for B^*, and $d_{A_1B_1}(X) = 0$, where $A_1 = A|_{\overline{\text{ranY}}} \in pk-QH$ and $B_1^* = B^*|_{\ker^{\perp}Y}$ is either p-hyponormal or injective dominant or an injective $pk-QH$ operator. In view of our remarks above, it follows from Theorem 2.1 that B_1 is normal and $d_{A_1^*B_1^*}(X) = 0$. Observe that $d_{A_1B_1}(X) = 0 = d_{A_1^*B_1^*}(X)$, B_1 normal, implies that A_1 is normal. It is not difficult to verify that invariant subspaces M of a $pk-QH$ operator A such that $A|_M$ is injective normal are reducing [9, Lemma 10]; hence $A = A_1 \oplus A_0$ for some operator A_0. But then $d_{A_1^*B_1^*}(X) = 0 \Longrightarrow d_{A^*B^*}(X) = 0$. □

Remark 2.5. **(i)**.The hypothesis that B^* is an injective dominant or an injective $pk-QH$ operator can not be relaxed in Theorem 2.4. Thus, let $A = B = N \oplus K$, where N is normal and K is k-nilpotent. Then A and $B^* \in pk-QH$. Let X_1 be any operator in the commutant of N, and let $X = X_1 \oplus K^{k-1}$. Then $\delta_{AB}(X) = 0$, but $\delta_{A^*B^*}(X) \neq 0$. Again, let $A = N \oplus K$, $X = 0 \oplus K^{k-1}$ and $B^* = D \oplus 0$ for some dominant operator D. Then $\delta_{AB}(X) = 0$, but $\delta_{A^*B^*}(X) \neq 0$.
(ii). If $A \in pk-QH$ is such that $A|_{\overline{\text{ran}A^k}}$ is normal, then $\overline{\text{ran}A^k}$ reduces A. To see this, let $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} \overline{\text{ran}A^k} \\ \ker A^{*k} \end{pmatrix}$, where A_{11} is normal.

Let $A_{11} = N \oplus 0$, where N is injective normal. Then [9, Lemma 10] implies that $A = \begin{pmatrix} N & 0 & 0 \\ 0 & 0 & A_2 \\ 0 & 0 & A_{22} \end{pmatrix} = N \oplus \begin{pmatrix} 0 & A_2 \\ 0 & A_{22} \end{pmatrix} = N \oplus A_0$. The operator A_0 is (of necessity) pk–quasihyponormal. It is easily verified that A_0 is $k+1$–nilpotent. Hence $\sigma(A_0) = \{0\}$. Applying [8, Corollary 6], it follows that A_0 is the direct sum of a normal with a nilpotent. Evidently, $A_2 = 0$.

Various combinations (such as $A \in pk - QH$ is injective and $B^* \in p - H$ (see [9, Theorem 11]) and variants (such as $A \in pk - QH$ and $B^* \in pk - QH$ such that $B^{*-1}(0)$ is reducing ($\Longrightarrow$ the pure part B^*_p of B^* is injective)) are possible in Theorem 2.4: we leave the formulation of such combinations to the reader. A version of Theorem 2.4 holds for $pk - QH$ operators A and spectral operators B. (See [5, Chapter XV] for information on spectral and scalar operators.)

Theorem 2.6. *If $\delta_{AB}(X) = 0$ (resp., $\triangle_{AB}(X) = 0$) for some operator $A \in pk - QH$, spectral operator $B \in B(\mathcal{K})$ such that $\overline{B\mathcal{K}} = \overline{B^k\mathcal{K}}$ and quasiaffinity X (resp., some operator $A \in pk - QH$ such that $0 \in \sigma(A) \Longrightarrow 0 \in \sigma_p(A)$, spectral operator B and quasiaffinity X), then A is normal, B is a scalar operator similar to A and $\delta_{A^*B^*}(X) = 0$ (resp., A is an invertible normal operator, B is a scalar operator similar to A and $\triangle_{A^*B^*}(X) = 0$).*

Proof. Case $\delta_{AB}(X) = 0$. The hypothesis $\overline{B\mathcal{K}} = \overline{B^k\mathcal{K}}$ implies that $B \in B(\overline{B\mathcal{K}} \oplus B^{*-1}(0))$ has a representation $B = B_{11} \oplus 0$, where B_{11} is spectral, and $X \in B(\overline{B\mathcal{K}} \oplus B^{*-1}(0), \overline{A^k\mathcal{H}} \oplus A^{*k^{-1}}(0))$ has a representation $X = \begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{pmatrix}$. Since $\overline{\text{ran}A^k\overline{\text{ran}X}} = \overline{\text{ran}X\overline{\text{ran}B^k}} = \overline{\text{ran}X\overline{\text{ran}B}}$, X_{11} is a quasiaffinity. Hence $\delta_{A_{11}B_{11}}(X_{11}) = 0 \Longrightarrow A_{11}$ is normal, B_{11} is a scalar operator similar to A_{11}, and $\delta_{A^*_{11}B^*_{11}}(X_{11}) = 0$ [7, Theorem 11]. Since X_{22} has dense range, $\delta_{AB}(X) = 0 \Longrightarrow A_{22}X_{22} = 0 \Longrightarrow A_{22} = 0$. Evidently, $A_{12} = 0$. (Observe that A_{11} is normal $\Longrightarrow \overline{\text{ran}A^k}$ reduces A; see Remark 2.5(ii).) Hence, $\delta_{A^*B^*}(X) = 0$, A is normal and B is a scalar operator similar to A.

Case $\triangle_{AB}(X) = 0$. Since X is a quasiaffinity, and since $0 \notin \sigma_p(A) \Longrightarrow 0 \notin \sigma(A)$, the hypothesis $\triangle_{AB}(X) = 0$ implies that A is an invertible p-hyponormal operator such that $\delta_{A^{-1}B}(X) = 0$. Applying [7, Theorem 11], it follows that A^{-1} is normal, B is a scalar operator similar to A^{-1} and $\delta_{A^{*-1}B^*}(X) = 0 = \triangle_{A^*B^*}(X)$. □

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